No Fine Theorem for Macrorealism: Limitations of the Leggett-Garg Inequality

Lucas Clemente* and Johannes Kofler†
Max Planck Institute of Quantum Optics, Hans-Kopfermann-Straße 1, 85748 Garching, Germany
(Received 9 September 2015; published 15 April 2016)

Tests of local realism and macrorealism have historically been discussed in very similar terms: Leggett-Garg inequalities follow Bell inequalities as necessary conditions for classical behavior. Here, we compare the probability polytopes spanned by all measurable probability distributions for both scenarios and show that their structure differs strongly between spatially and temporally separated measurements. We arrive at the conclusion that, in contrast to tests of local realism where Bell inequalities form a necessary and sufficient set of conditions, no set of inequalities can ever be necessary and sufficient for a macrorealistic description. Fine’s famous proof that Bell inequalities are necessary and sufficient for the existence of a local realistic model, therefore, cannot be transferred to macrorealism. A recently proposed condition, no-signaling in time, fulfills this criterion, and we show why it is better suited for future experimental tests and theoretical studies of macrorealism. Our work thereby identifies a major difference between the mathematical structures of local realism and macrorealism.

DOI: 10.1103/PhysRevLett.116.150401

The violation of classical world views, such as local realism [1] and macrorealism [2,3], is one of the most interesting properties of quantum mechanics. Experiments performed over the past decades have shown violations of local realism in various systems [4–6], while violations of macrorealism are on the horizon [7–24]. The latter endeavors pave the way towards the experimental realization of Schrödinger’s famous thought experiment [25]. In the future, they might offer insight into important foundational questions, such as the quantum measurement problem [26], and allow experimental tests of (possibly gravitational) extensions of quantum mechanics [27].

Historically, the discussion of tests of macrorealism (MR) follows the discussion of tests of local realism (LR) closely: Leggett-Garg inequalities (LGIs) [2] are formulated similarly to Bell inequalities [1,28,29], and some concepts, e.g., quantum contextuality [30], are connected to both fields [31–35]. However, recently, a discrepancy between LR and MR has been identified: Whereas Fine’s theorem states that Bell inequalities are both necessary and sufficient for LR [36], a combination of arrow of time (AoT) and no-signaling in time (NSIT) [37] equalities are necessary and sufficient for the existence of a macrorealistic description [38]. A previous study [38] also demonstrated that LGIs involving temporal correlation functions of pairs of measurements are not sufficient for macrorealism, but did not rule out a potential sufficiency of other sets of LGIs, e.g., of the Clauser-Home (CH) type [29,39], leaving open the possibility of a Fine theorem for macrorealism. Moreover, cases have been identified where LGIs hide violations of macrorealism [31] that are detected by a simple NSIT condition [37]. The latter fails for totally mixed initial states, where a more involved NSIT condition is required [38]. These fundamental differences between tests of local realism and macrorealism seem connected to the peculiar definition of macrorealism [40,41].

In this Letter, we analyze the reasons for and the consequences of this difference. We show that the probability space spanned by quantum mechanics (QM) is of a higher dimension in a MR test than in a LR test, and we analyze the resulting structure of the probability polytope. We conclude that inequalities—excluding the pathological case of inequalities pairwise merging into equalities—are not suited to be sufficient conditions for MR, and they form only weak necessary conditions. Fine’s theorem [36], therefore, cannot be transferred to macrorealism (unless one uses potentially negative quasiprobabilities [42]). Our study thus identifies a striking difference between the mathematical structures of LR and MR. While current experimental tests of macrorealism overwhelmingly use Leggett-Garg inequalities, this difference explains why NSIT is better suited to be a witness of nonclassicality; i.e., why it is violated for a much larger range of parameters [37,38].

Let us start by reviewing the structure of the LR polytope (LR), as described in Refs. [43–45]. Consider a LR test between $n \geq 2$ parties $i \in \{1,\ldots,n\}$. Each party can perform a measurement in one of the $m \geq 2$ settings $s \in \{1,\ldots,m\}$. Each setting has the same number $\Delta \geq 2$ of possible outcomes $q \in \{1,\ldots,\Delta\}$, and, to allow for all possible types of correlations, it may measure a distinct property of the system. We can define probability distributions $p_{q_{1},\ldots,q_{n}|s_{1},\ldots,s_{n}}$ for obtaining outcomes $q_{1},\ldots,q_{n}$, given the measurement settings $s_{1},\ldots,s_{n}$. If a party $i$ chooses not to perform a measurement, the corresponding “setting” is labeled $s_{i} = 0$, and there is only one “outcome” labeled $q_{i} = 0$ (e.g., $p_{q_{i}|s_{i}=0}$ when only the first party performs a measurement). We leave out final zeros, e.g., $p_{q_{1},\ldots,q_{n},0,\ldots,0|s_{1},\ldots,s_{n},\ldots,0} = p_{q_{1},\ldots,q_{n}|s_{1},\ldots,s_{n}}$. Note that this convention differs from the literature for LR tests, where the case of no measurement is often left out [43,45], but simplifies the comparison between LR and MR tests. Each
There are measurements on LR test panel of Fig. 1. Shimony-Holt inequalities for arrive at dimension [43] NS polytope to a NS polytope (These in fact, depend on bounds [46], the space of probability distributions from quantum mechanics obeys NS, and due to Tsirelson contributions, it can be seen as a point in a subspace and solely delimited by positivity constraints. The resulting subspace is called the probability polytope $P$. In a LR test with spacelike separated parties, special relativity prohibits signaling from every party to any other, $\forall i, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n, s_1, \ldots, s_n, s_i \neq 0$:

$$p_{q_1, \ldots, q_n, s_1, \ldots, s_n} = \sum_{q_i=1}^{\Delta} p_{q_1, \ldots, q_i, s_1, \ldots, s_n}.$$ 

These no-signaling (NS) conditions restrict the probability polytope to a NS polytope (NS) of lower dimension. Taking their linear dependence, both amongst each other and with the normalization conditions, into account, we arrive at dimension [43]

$$\dim \text{NS} = n(\Delta - 1) + 1 - 1.$$

Since quantum mechanics obeys NS, and due to Tsirelson bounds [46], the space of probability distributions from spatially separated experiments implementable in quantum mechanics, $\text{QMS}_S$, is located strictly within the NS polytope. Furthermore, the space of local realistic probability distributions, LR, is a strict subspace of $\text{QMS}_S$. It is delimited by Bell inequalities (e.g., the CH and Clauser-Horne-Shimony-Holt inequalities for $n = m = \Delta = 2$) and positivity conditions, and it therefore forms a polytope within $\text{QMS}_S$ [36,43]. In summary, we have $P \supset \text{NS} \supset \text{QMS}_S \supset \text{LR}$, with $\dim P > \dim \text{NS} = \dim \text{QMS}_S = \dim \text{LR}$. The structure of the NS, QMS, and LR spaces is sketched in the left panel of Fig. 1.

In a test of MR, temporal correlations take the role of a LR test’s spatial correlations. Instead of spatially separated measurements on $n$ systems by different observers, a single observer performs $n$ sequential (macroscopically distinct) measurements on one and the same system. Again, each measurement is either skipped (“0”) or performed in one of the $m \geq 1$ [47] settings, with $\Delta$ possible outcomes each. With this one-to-one correspondence, the resulting probability polytope $P$ in the space $\mathbb{R}^{(m\Delta + 1)n - (m + 1)^n}$ is identical to the one in the Bell scenario. However, without further physical assumptions, no-signaling in temporally separated experiments is only a requirement in one direction: While past measurements can affect the future, causality demands that future measurements cannot affect the past. This assumption is captured by the AoT conditions:

$$\forall i \geq 2: \forall q_1, \ldots, q_{i-1}, s_1, \ldots, s_{i-1}, \text{ with } \Sigma_{j=1}^{i-1} s_j \neq 0, s_i \neq 0:\nabla p_{q_1, \ldots, q_{i-1}, s_1, \ldots, s_{i-1}} = \sum_{q_i=1}^{\Delta} p_{q_1, \ldots, q_i, s_1, \ldots, s_i}.$$ 

Counting the number of equalities in Eq. (6) shows that their number is

$$\sum_{i=2}^{n} [(m\Delta + 1)^{i-1} - 1]m = \frac{(m\Delta + 1)^n - nm\Delta - 1}{\Delta},$$

FIG. 1. (Left panel) A sketch of subspaces in a LR test [45]. The no-signaling polytope (NS) contains the space of probability distributions realizable from spatially separated experiments in quantum mechanics (QMS), which contains the local realism polytope (LR). LR is delimited by Bell inequalities and positivity conditions. NS, QMS, and LR have the same dimension. A Bell inequality (BI) is also sketched, delimiting LR. Another tight Bell inequality (BI') is less suited to be a witness of non-LR behavior, and it illustrates the role of Leggett-Garg inequalities in macrorealism tests. (Right panel) A sketch of polytopes in a MR test. The arrow of time polytope (AoT) is equal to the space of probability distributions realizable from temporally separated experiments in quantum mechanics (QMT), which contains the macrorealism polytope (MR). MR is a polytope of lower dimension, located fully within the QMT subspace and solely delimited by positivity constraints. Since each probability can easily be minimized or maximized individually, MR reaches all facets of AoT. A Leggett-Garg inequality (LGI) is also sketched; it is a hyperplane of dimension $\dim \text{QMT} - 1$, which, in general, is much larger than $\dim \text{MR}$. Note that the LGI can only touch MR (i.e., be tight) at the boundary of the positivity constraints.
where the first factor in the sum counts the setting and outcome combinations for times \(1, \ldots, i - 1\), excluding the choice of all \(s_j = 0\), and the second factor the number of settings at time \(i\). All listed conditions are linearly independent due to their hierarchical construction; see Fig. 2. However, a number of the normalization conditions for the marginal distributions, already subtracted in Eq. (2), are not linearly independent from AoT, and thus they become obsolete. Their number is obtained by counting the different settings in Eq. (6):

\[
\sum_{i=2}^{n} [(m + 1)^{i-1} - 1] m = (m + 1)^n - nm - 1. \tag{8}
\]

The remaining normalization conditions are the ones for probability distributions with just one measurement and for the “0 distribution”; there are \(nm + 1\) such distributions. Taking Eq. (2), subtracting Eq. (7), and adding Eq. (8), we conclude that the AoT conditions restrict the probability polytope to an AoT polytope (AoT) of dimension

\[
\dim \text{AoT} = \frac{[(m\Delta + 1)^n - 1](\Delta - 1)}{\Delta}. \tag{9}
\]

By simple extension of the proof in Ref. [38], the set of all NSIT conditions,

\[
\forall i < n, q_1, q_{i-1}, q_{i+1}, \ldots, q_m, s_1, \ldots, s_n; \sum_{j \neq i} s_j \neq 0, s_i \neq 0:
\]

\[
P_{q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_m; s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n} = \frac{\Delta}{\sum_{q_i}} P_{q_1, \ldots, q_i, q_{i+1}, \ldots, q_m; s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_n}, \tag{10}
\]

is, together with AoT, necessary and sufficient for macrorealism. To get from AoT to the macrorealism polytope, MR, we therefore require a linearly independent subset of these conditions. However, since the AoT conditions from Eq. (6) plus the NSIT conditions from Eq. (10) are equivalent to the NS conditions from Eq. (4), we arrive at MR with the same dimension as the LR polytope:

\[
\dim \text{MR} = \dim \text{LR} = (m\Delta - 1 + 1)^n - 1. \tag{11}
\]

We are left with the question of how the space of probability distributions realizable from temporally separated experiments in quantum mechanics, QMT, relates to AoT. Fritz has shown in Ref. [48] that QMT = AoT for \(n = m = \Delta = 2\), if we allow for positive-operator valued measurements (POVMs). Let us now generalize his proof to arbitrary \(n, m, \Delta\)'s. We do so by constructing a quantum experiment that produces all possible probability distributions which are allowed by AoT.

Consider a quantum system of dimension \((m\Delta + 1)^n\), with states enumerated as \(|q_1, \ldots, q_n; s_1, \ldots, s_n\rangle\). As with the probability distributions, final zeros may be omitted. The initial state of the system is \(|0\ldots0; 0\ldots0\rangle\). Now, \(n\) POVMs are performed on the system. The measurements are chosen such that, depending on their setting and outcome, they take the system to the corresponding state: Performing a measurement on a system in state \(|q_1, \ldots, q_{i-1}; s_1, \ldots, s_{i-1}\rangle\) with setting \(s_i\) and obtaining outcome \(q_i\) should leave the system in state \(|q_1, \ldots, q_i; s_1, \ldots, s_i\rangle\). This is accomplished by choosing Kraus operators for the \(i\)th measurement in basis \(s_i\) for outcome \(q_i\) to be
For $i = 1$, the first sum in Eq. (12) reduces to the single term $\sqrt{P_{q_i|s_1}}\ket{q_1; s_1}\bra{00000}$, while the second sum remains unchanged. The second sum in Eq. (12) is necessary for the completeness relation $\sum_{q_i} (K_{i,q_i}^{i}) \dagger K_{i,q_i}^{i} = 1$. The above definitions also work for $s_i = 0$, where $r_{q_i=0|q_i, \ldots, q_{i-1}, s_i, \ldots, s_{i-1}} = 1$, and $(K_{i,q_i}^{i}) \dagger K_{i,q_i}^{i} = 1$. The conditional probabilities $r$ in Eq. (12) can be obtained from the probabilities $p$ using the assumption of AoT:

$$
K_{i,q_i}^{i} = \sum_{s_i, \ldots, s_{i-1}, q_i, \ldots, q_{i-1}} \sqrt{r_{q_i|q_i, \ldots, q_{i-1}, s_i, \ldots, s_{i-1}}} |q_i, \ldots, q_{i-1}, s_i, \ldots, s_{i-1} \rangle \langle q_i, \ldots, q_{i-1}, s_i, \ldots, s_{i-1}| + \sum_{q_i, \ldots, q_{i-1}} \frac{1}{\sqrt{\Delta}} |q_i, \ldots, q_n, s_i, \ldots, s_n \rangle \langle q_i, \ldots, q_n, s_i, \ldots, s_n|.
$$

(12)

This construction gives a recipe to obtain any point in the AoT probability space in a quantum experiment. We have therefore shown that AoT = QM$_T$ for any choice of $n$, $m$, $\Delta$.

Note that the probability distributions constructed above can also be achieved by a purely classical stochastic model, albeit with invasive measurements. Such an experiment would not therefore convince a macrorealist to give up his or her worldview. For that to happen, an experiment needs to properly address the clumsiness loophole [2,49,50]. The relevant methods previously established for the LGI can also be applied to NSIT-based experiments [24].

Since AoT is a polytope, QM$_T$ with POVMs is also a polytope, and no nontrivial Tsirelson-like bounds exist. If, on the other hand, we only allowed projective measurements, we would have QM$_T$ $\subset$ AoT with nontrivial Tsirelson-like bounds, as shown in Ref. [48]. In this case, QM$_T$ would not be a polytope. It is easy to see that QM with projectors is unable to reproduce some probability distributions: $n = 2, m = 1, \Delta = 2, p_{11|11} = 1, p_{01|01} = 0$ fulfills AoT but cannot be constructed in projective quantum mechanics since the initial state must be an eigenstate of the first measurement. Here, we consider the general case of POVMs.

In summary, we have

$$
P \supset NS \supset QM_S \supset LR \cong QM_T \supset MR
$$

with $NS = MR$, and dimensions

$$
dim P > dim NS = dim QM_S = dim LR \cong QM_T > dim MR
$$

The structure of AoT, QM$_T$ and MR within $P$ is sketched on the right side of Fig. 1, the dimensions of all of the mentioned subspaces are printed in Table I

Finally, let us compare the characteristics of quantum mechanics in LR and MR tests. Trivially, QM fulfills NS between spatially separated measurements, and AoT between temporally separated measurements [51]. While QM$_S$ and LR have the same dimension and are separated by Bell inequalities, QM$_T$ and MR span subspaces with different dimensions. Inequalities can never reduce the dimension of the probability space since they act as a hyperplane separating the fulfilling from the violating volume of probability distributions. We conclude that no combination of (Leggett-Garg) inequalities can be sufficient for macrorealism.

The observation that inequalities cannot be sufficient for macrorealism and the differences in the structure of the probability space shown above present fundamental discrepancies between LR and MR. Fine’s observation [36] that Bell inequalities are necessary and sufficient for LR can therefore not be transferred to the case of LGIs and MR. More precisely, Fine’s proof uses the implicit assumption of NS, which is obeyed by all reasonable physical theories, including QM. However, the temporal analogue to NS is the conjunction of AoT and NSIT, where AoT is obeyed by all reasonable physical theories, while NSIT is violated in QM. Therefore,

| TABLE I. Dimensions of the probability space $P$ and its subspaces reachable by spatially separated (QM$_S$) or temporally separated (QM$_T$) experiments in quantum mechanics, local realism (LR), and macrorealism (MR). There are $n$ spatially or temporally separated measurements with $m$ settings and $\Delta$ outcomes each. |
|-----------------|-----------------|-----------------|-----------------|
|                | LR test         | MR test         |
| Number of unnormalized distributions | $(m\Delta + 1)^n$ | $(m\Delta + 1)^n$ |
| $dim P$        | $dim NS$        | $dim QM_T$      |
| $dim QM_S$, $dim QM_T$ | $m(\Delta - 1 + 1)^n - 1$ | $< (m\Delta + 1)^n - 1$ |
| $dim LR$, $dim MR$ | $[m(\Delta + 1)^n - 1](\Delta - 1)/\Delta$ | $- 1$ |
where “BIs” and “LGIs” denote the sets of all Bell and Leggett-Garg inequalities, respectively.

Moreover, since MR is a polytope with a smaller dimension than QM, LGIs can only touch MR (i.e., be tight) at one facet, i.e., a positivity constraint, as sketched in Fig. 1 on the right side. A comparable Bell inequality, sketched in Fig. 1 on the left as B’, clearly illustrates the limitations resulting from this requirement. In an experimental test of MR, using a LGI, therefore, needless restricts the parameter space where violations can be found. The favorable experimental feasibility of NSIT is demonstrated by the theoretical analyses of Refs. [37,38], as well as the recent experiment of Ref. [24]. Note also the mathematical simplicity of the NSIT conditions when compared to the LGI. We conclude that, for further theoretical studies and future experiments, it might be advantageous to eschew the LGIs and rather use NSIT.

We acknowledge support from the EU Integrated Project SIQS.

---

1lucas.clemente@mpq.mpg.de  
2johannes.kofler@mpq.mpg.de  

[47] In contrast to LR tests, where m ≥ 2 is required to observe quantum violations, m = 1 allows for violations of MR and is in fact the most considered case in the literature.  
[51] To show that QM fulfills NS, we consider a setup with only two parties, 1 and 2, performing measurements with POVM elements ̂Mq1,q2; ̂ Mq1,q2, respectively, on a
two-particle state $\hat{\rho}_{12}$. We then calculate
\[
\sum_{q_1} p_{q_1} \rho_{q_1} = \sum_{q_1} \text{tr}[\hat{M}_{q_1}^+ \hat{M}_{q_1} \hat{\rho}_{12}] = \text{tr}[\hat{M}_{q_1}^+ \hat{M}_{q_1} \hat{\rho}_{12}] = \rho_{q_1} = \rho_{q_1}.
\]
To show that QM fulfills AoT, we consider a setup where $\hat{M}_{q_1}^+ \hat{M}_{q_1}$ are measured at time 1 on state $\hat{\rho}_{1}$, and $\hat{M}_{q_2}^+ \hat{M}_{q_2}$ are measured at time 2. We then have
\[
\sum_{q_2} p_{q_2} \rho_{q_2} = \sum_{q_2} \text{tr}[\hat{M}_{q_2}^+ \hat{M}_{q_2} \rho_{q_2}] = \text{tr}[\hat{M}_{q_2}^+ \hat{M}_{q_2} \rho_{q_2}] = \rho_{q_2} = \rho_{q_2}.
\]