Ludwig-Maximilians-Universität München
Technische Universität München
Max-Planck-Institut für Quantenoptik

Quantum Information with Fermionic Gaussian States

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Ludwig-Maximilians-Universität München
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1st reviewer: Prof. Dr. Ignacio Cirac

2nd reviewer: Dr. Géza Giedke

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Eliška Greplová

Abstract

The subject of this Thesis is quantum information with fermions in the Gaussian setting. We study finite-dimensional fermionic Gaussian quantum states and channels. We show that Gaussian states are extremal with respect to a certain class of functionals among all states with the same second moments. This result can be used to bound certain entanglement-like quantities as well as to compute the quantum capacities of degradable fermionic channels. To apply this result, we derive a standard form of fermionic Gaussian channels and provide criteria for their degradability. We point out the applications of fermionic Gaussian states in quantum computation theory as well as recent results towards the realization of fermionic channels for which the presented formalism may become relevant. The Thesis is supplemented by an Appendix in which we explain the implications of the parity superselection rule for quantum information processing at the example of quantum teleportation.

Contents

Chapter 1

Introduction

In this Thesis we study finite-dimensional fermionic Gaussian states and channels. In physics, Gaussian approximation is a frequently used tool for solving many-body problems. Gaussian approximation relies on describing system fully in terms of two-point correlation functions. This means that all the higher order correlations can be expressed as a product of two-point correlation functions, which radically reduces the complexity of quantum mechanical description.

In context of quantum information Gaussian states have been successfully used to describe the states of light, i.e. bosonic systems. Given the recent progress in control and manipulation of fermionic systems we can think of solid state systems being employed in experimental realization of quantum information processing. Inspired by the results achieved using Gaussian description in bosonic setting we explore its impact within fermonic systems.

In this work we review the possible descriptions of fermionic Gaussian states. We prove the theorem concerning extremality of Gaussian states with respect to certain class of functionals. The extremality properties can be used to estimate some entanglement-like quantities as well as capacities of quantum channels. We study Gaussian transformations, i.e. maps transforming Gaussian states into Gaussian states. We derive a standard form of fermionic Gaussian transformations as well as a standard form of fermionic Gaussian channels, which can be employed to characterize the degradability of the fermionic Gaussian channels.

The characterization and comparison of the existing descriptions of fermionic Gaussian states are given in the Chapters 2 and 3. In addition to that we provide the maps that translate one representation into another.

Chapter 4 features the proof of the Extremality theorem for fermionic Gaussian states together with an outline of the possible applications.

In Chapter 5 we firstly give an overview of the possible descriptions of quantum channels together with the general introduction into the capacity theory. We explain the notion of degradability and its effect on the capacity evaluation. Then we give the derivation of the simple form of the fermionic Gaussian maps and channels. We conclude with the derivation of the conditions on degradability of fermionic Gaussian channels.

In Chapter 6 we describe the convex hull of fermionic Gaussian states and its application in quantum computation theory. In order to complete the picture from a computational point of view we also give an overview of the recent results of the classical simulatability of fermionic Gaussian system and the options for realizing universal quantum computation in such systems.

As pointed out at the beginning of this paragraph there were many impressive results with bosonic Gaussian framework. When trying to translate them to fermionic world one has to be aware of a crucial difference and that is so-called parity superselection rule. All the processes that have been observed so far in nature have the feature of preserving parity of the system. This leads to many complications especially when one wants to rephrase bosonic results. To illustrate this difference that is emphasized many times within the Thesis we complement the text by appendix illustrating how to include parity super selection rule into the simple protocol that is however crucial for quantum information processing.

To conclude the Thesis in Chapter 7 we give a short outlook of the systems that are or could be potentially useful as a platform for experimental realization of the framework introduced in this work.

Chapter 2

Fermionic Gaussian states: Covariance Matrix Approach

2.1 Toolbox

In this section we aim to introduce neccesary notation and review basic notions connected to quantum description of half-integer spin particles: firstly we antisymmetrize the wavefunctions, secondly we build up the Fock space representation of fermionic states and finally we introduce an algebra of relevant observables.

2.1.1 Anti-symmetric tensor product

Let us consider one-particle Hilbert space $\mathcal{H}$ of dimension d. The well-known tensor product operation $\otimes$ is postulated to describe composite systems [65]. Let us consider the states $\Psi_i \in \mathcal{H}_i$, where $i \in \{1, 2, \dots N\}$. Then the state of composite system on N-fold product Hilbert space $\bigotimes_i \mathcal{H}_i$ will be described by the object $\Psi_1 \otimes \Psi_2 \otimes \cdots \otimes \Psi_N$. But we must not forget that when working with fermions the resulting wavefunction has to be antisymmetric. Antisymmetrized version of the tensor product $\otimes$ will be denoted by $\wedge$ and defined as follows

$$\Phi_1 \wedge \cdots \wedge \Phi_k = \frac{1}{\sqrt{k!}} \sum_\sigma \epsilon(\sigma) \Phi_{\sigma(1)} \otimes \cdots \otimes \Phi_{\sigma(k)}, \tag{2.1}$$

where the summation is over all permutations σ of the k indeces and $\epsilon(\sigma) = \pm 1$ (according to the parity of the permutation).

2.1.2 Fermionic Fock space

Limited by the Pauli exclusion principle we can not arbitrarily increase degrees of freedom of the system by adding more particles. In particular, having n-dimensional one-particle Hilbert space (fermion with n modes) and adding one more (indistinguishable) fermion the available number of states will be $\binom{n}{2}$. Analogously we conclude that when adding k fermions, the number of states will be

$\binom{n}{k}$. It turns out that the appropriate representation that captures creation and annihilation of the particles in the system is the so-called Fock representation describing the state via the particle number in the given modes. More precisely, the Fock space built on one-particle Hilbert space $\mathcal{H}$ is defined as

$$\mathcal{F}(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^{(2)} \oplus \cdots \oplus \mathbb{C}, \tag{2.2}$$

where the first term in the sum corresponds to fermionic vacuum, the last one to the completely filled Fermi sea, $\mathcal{H}$ to the one-particle, $\mathcal{H}^{(2)}$ two particles and so on. The dimension of $\mathcal{F}(\mathcal{H})$ is $2^{\dim\mathcal{H}}$.

2.1.3 CAR algebra

One way how to build fermionic Fock space starting from one-particle Hilbert space $\mathcal{H}$ is to introduce linear operator $a^\dagger(\Phi)$ for every vector $\Phi \in \mathcal{H}$ such that $a^\dagger(\Phi) : \mathcal{H}^{(k)} \to \mathcal{H}^{(k+1)}$. We can introduce the action of the operator via

$$a^\dagger(\Phi)(\Psi_1 \wedge \cdots \wedge \Psi_k) = \Phi \wedge \Psi_1 \wedge \cdots \wedge \Psi_k \tag{2.3}$$

As we can use the operator $a^\dagger$ to 'create' particle in the system we can define an adjoint operator that will annihilate one particle using an linear operator $a : \mathcal{H}^{(k)} \to \mathcal{H}^{(k-1)}$:

$$a(\Phi)(\Psi_1 \wedge \cdots \wedge \Psi_k) = \langle \Phi, \Psi_1 \rangle \Psi_2 \wedge \cdots \wedge \Psi_k \tag{2.4}$$

Annihilation and creation operators a and $a^\dagger$ fulfill the so-called canonical anti-commutation relations (CAR)

$$\{a(\Phi), a(\Psi)\} = 0 \qquad \{a(\Phi), a^\dagger(\Psi)\} = \langle \Phi, \Psi \rangle \mathbb{1}, \tag{2.5}$$

where the anticommutator is defined as $\{A, B\} = AB + BA$. The algebra of creation and annihilation operators in called an algebra of canonical anti-commutation relations (CAR algebra) and coincides with the algebra of linear transformations on Fock space (2.2). For a more detailed description of the construction outlined above, see [12].

2.1.4 Notation

We fix the basis of the one-particle Hilbert space $\{\Phi_i\}$ and introduce the notation:

$$a_i = a(\Phi_i), \quad a_i^\dagger := a^\dagger(\Phi_i), \quad \forall i. \tag{2.6}$$

We can then rewrite the canonical anticommutation relations as (2.5) as follows

$$\{a_k, a_l\} = \{a_k^\dagger, a_l^\dagger\} = 0, \quad \{a_k, a_l^\dagger\} = \delta_{kl}. \tag{2.7}$$

Let us also introduce hermitian Majorana modes of the system that are defined

$$c_{2j-1} = a_j^\dagger + a_j, \quad c_{2j} = (-i)(a_j^\dagger - a_j). \tag{2.8}$$

In this case canonical anticommutation relations take the compact form

$$\{c_k, c_l\} = 2\delta_{kl}. \tag{2.9}$$

The algebra formed by the operators $\{c_i\}_{i=1}^{2n}$ is called *Clifford algebra*.

Transformation in between fermionic and Majorana opearators is achieved by matrix of the block form

$$\Omega = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ i\mathbb{1} & -i\mathbb{1} \end{pmatrix}. \tag{2.10}$$

In particular, the vector of fermionic operators ordered as $\vec{a}^T = (a_1, a_2, \dots, a_1^\dagger, a_2^\dagger \dots)$ is mapped on the vector of Majorana operators ordered as $\vec{c}^T = (c_1, c_3, \dots, c_2, c_4, \dots)$: $\Omega\vec{a} = \vec{c}$.

2.2 Physical motivation

In many areas of physics one has to has to face solving quantum many body problems, which is often a computationally difficult if not impossible task. It is also well known fact that under appropriate approximations complicated Hamiltonians with many-body interactions can be often mapped onto Hamiltonians that are quadratic in annihilation and creation operators and have the generic form [11]

$$H = \sum_{ij} C_{ij} a_i^\dagger a_j + \sum_{ij} (A_{ij} a_i^\dagger a_j^\dagger + \text{h.c.}), \tag{2.11}$$

where $a_i, a_i^\dagger$ are fermionic annihilation and creation operators (see (2.6)). Hamiltonians of this class are diagonalized using so-called Bogoliubov transformation that maps fermionic creation and annihilation operators on the creation and annihilation operators of non-interacting quasi-particles. More concretely

$$a_i \mapsto \gamma_i q_i + \kappa_i q_i^\dagger \tag{2.12}$$

$$a_i^\dagger \mapsto \bar{\gamma}_i q_i^\dagger + \bar{\kappa}_i q_i, \tag{2.13}$$

where γ_i, κ_i are complex numbers such that $\kappa_i^2 + \gamma_i^2 = 1$, and q_i, $q_i^\dagger$ are annihilation and creation operators for the quasi-particles that are expressible as a linear combination of the creation and annihilation operators $a^\dagger, a$. After such transformation, the Hamiltonian (2.11) is of the diagonal form

$$H = \epsilon_k q_k^\dagger q_k. \tag{2.14}$$

Many models of high physical relevance are diagonalizable using Bogoliubov transformation. Let us mention for example Hubbard models or mean-field theory in the first approximation of BCS superconductivity theory.

The ground state of quadratic Hamiltonian (2.11) that describe definite set of quasi-particle excitations belong to the class of so-called fermionic Gaussian

states (and in case of non-degenerate spectrum every eigenstate is of that type). These states are fully characterized by the second order correlations, because all the higher moments factorize. This statement is called Wick's theorem. More precisely, let us define

$$c(\mathbf{x}) = c_1^{x_1} c_2^{x_2} \dots c_{2n}^{x_{2n}}, \tag{2.15}$$

where $c_i, i \in \{1, 2, \dots 2n\}$ are Majorana fermions defined via (2.8)
and $\mathbf{x} = (x_1, x_2, \dots, x_{2n})$ is a binary string of $2n$ bits. Then the statement of the theorem is [14]

Theorem 2.1. *For every Gaussian state ρ and even binary string $\mathbf{x} \in \{0,1\}^{2n}$ with the weight $|x| = 2l$ one has*

$$\mathrm{Tr}(\rho c(\boldsymbol{x})) = i^l \mathrm{Pf}(\Gamma[\mathbf{x}]), \tag{2.16}$$

where $M[\mathbf{x}]$ is a submatrix of the so-called covariance matrix that contains all second moments of the state ρ. Γ is defined via $\Gamma_{pq} = \frac{i}{2}\mathrm{Tr}(\rho[c_p, c_q])$ and $\Gamma[\mathbf{x}]$ is obtained by choosing those matrix elements of Γ for which $x_p = x_q = 1$. For the binary string with the odd weight $|x| = 2l + 1$ we have

$$\mathrm{Tr}(\rho c(\boldsymbol{x})) = 0. \tag{2.17}$$

Pf in he equation (2.16) stands for Pfaffian, which is a polynomial in the matrix entries. For the antisymmetric matrices (that will be exactly of our interest) there is a simple connection between Pfaffian and determinant, namely

$$\mathrm{Pf}(A)^2 = \det(A), \tag{2.18}$$

for every antisymmetric matrix A.

In the following sections we further discuss the covariance matrix approach to the description of fermionic Gaussian states. There is a closely related method of description, the so-called symbol formalism. We will review the main features of both of these and find the map that translates between covariance matrix and symbol formalism.

2.3 Fermionic Gaussian states: Covariance Matrix Approach

As motivated above we will define Gaussian states as follows

Definition 2.1. *Gaussian states are those satisfying Wick theorem 2.1.*

Equivalenty we can state that Gaussian states are fully characterized by their second moments. In the fermionic settings we can therefore write the density matrix of the Gaussian state in the form [62]

$$\rho = K \cdot \exp\left[-\frac{i}{4} c^T G c\right] \tag{2.19}$$

with $c = (c_1, \ldots, c_{2n})$ being a vector of Majorana operators (2.8), K normalization constant and G real anti-symmetric $2n \times 2n$ matrix. Every anti-symmetric matrix can be brought into a block diagonal form using the group of special orthogonal transformations SO($2n$):

$$OGO^T = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & -\beta_j \\ \beta_j & 0 \end{pmatrix} \qquad \text{with} \qquad O \in \mathrm{SO(2n)}. \tag{2.20}$$

The β_j's are called the Williamson eigenvalues of the matrix G. Block diagonalization of G also leads to so-called standard form of fermionic Gaussian state

$$\rho = \frac{1}{2^n} \prod_{k=1}^{n} (\mathbb{1} + i\lambda_k \tilde{c}_{2k-1} \tilde{c}_{2k}), \tag{2.21}$$

where $\tilde{c} = O^T c$ and λ_k's real numbers from the interval$[-1, 1]$.The standard form of Gaussian state has following physical meaning. The unitary operation needed to transform the Gaussian state into the standard form is to decouple the different modes and translate it into a state that is a product of independent thermal states diagonal in the number basis (Fock basis). The coefficients λ_i are therefore related to the temperature of the system.

As it has been already suggested in the discussion above it appears to be beneficial to order second moments fully characterizing the state in the form of the covariance matrix Γ:

$$\Gamma_{kl} = \frac{i}{2} \mathrm{Tr}(\rho[c_k, c_l]). \tag{2.22}$$

This matrix can of course also be brought into the block diagonal form via orthogonal transformation

$$O\Gamma O^T = \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & -\lambda_j \\ \lambda_j & 0 \end{pmatrix} \qquad \text{with} \qquad O \in \mathrm{SO}(2n). \tag{2.23}$$

It is easy to see that the actual eigenvalues of the covariance matrix Γ can be expressed in terms of Williamson eigenvalues as $\pm i\lambda_j$. As pointed out in [62] Williamson eigenvalues also determine the connection in between covariance matrix Γ and the matrix G from the definition (2.19), in particular $\lambda_j = \tanh(\frac{\beta_j}{2})$. Using positivity and normalization of the state we can conclude that $\lambda_j \in [-1, 1], \forall j$ and therefore every admissible covariance matrix has to fulfill

$$i\Gamma \leq 1 \tag{2.24}$$

or equivalently

$$\Gamma\Gamma^\dagger \leq 1. \tag{2.25}$$

To conclude the list of basic properties let us introduce the covariance matrix criterion of the purity of the state. The Gaussian state is pure if and only if its covariance matrix meets $\Gamma^2 = -\mathbb{1}$ (for a review of purity of the state criteria in the language of covariance matrices see [3]). Let us also note that in the terms of standard form purity corresponds to $\lambda_k = \pm 1, \forall k$, which agrees with the thermal state intuition suggested above.

2.4 Fermionic Gaussian State: Symbol Approach and Covariance Matrix Comparison

There is an important subclass of fermionic Gaussian states, the so-called gauge-invariant states, that commute with the particle number operator. They have been studied in particular detail [31]. Here the definition reads:

Definition 2.2. *A linear functional ω on the CAR algebra that assigns zero values to all monomials in creation and annihilation operators except for:*

$$\omega(a^\dagger(\Phi_1)\dots a^\dagger(\Phi_k)a(\Psi_l)\dots a(\Psi_1)) = \delta_{kl}\mathrm{Det}[\langle\Psi_i,\kappa\Phi_j\rangle]_{ij} \tag{2.26}$$

extends to a states on Fock space if and only if the linear one-particle transformation κ satisfies $0 \leq \kappa \leq 1$.

For gauge-invariant states there is very straightforward way how to express density matrix, important thermodynamic quantities and maps relevant in quantum information theory [31]. Now we are interested in the comparison of covariance matrix and symbol and finding the respective translational maps that will help us to translate results obtained in different frameworks.

Using notation (2.6) we see that matrix elements of the symbol (2.26) are

$$\kappa_{ij} = \langle a_i^\dagger a_j\rangle = \mathrm{Tr}(\rho a_i^\dagger a_j), \tag{2.27}$$

where ρ is the density matrix of the state. Comparing this to (2.22) we see that while covariance matrix (2.22) contains all possible combinations of the Majorana modes of the system, symbol only has the terms of the form $a^\dagger a$, which also means that the covariance matrix has twice as large dimension as the symbol.

The generalization of the symbol is then straightforward. We will call the generalized symbol Λ and its matrix elements will be the expectation values of the state on the all possible quadratic combinations creation and annihilation operators characterizing fermionic system:

Definition 2.3. *The symbol of n-mode fermionic state is a 2n×2n complex matrix with the elements*

$$\Lambda_{ij} = \langle a_i^\# a_j^\#\rangle = \mathrm{Tr}(\rho a_i^\# a_j^\#), \tag{2.28}$$

where # denotes either † or nothing.

There is a freedom left in ordering of $a_i^\# a_j^\#$ terms, we will choose

$$\Lambda = \begin{pmatrix} \langle a_k^\dagger a_l\rangle & \langle a_m^\dagger a_n^\dagger\rangle \\ \langle a_p a_q\rangle & \langle a_u a_v^\dagger\rangle \end{pmatrix}. \tag{2.29}$$

We also see that covariance matrix elements are evaluated on the commutators of Majorana operators. So we first define an affine map ϵ that maps Λ to $\tilde{\Lambda}$ defined as

$$\tilde{\Lambda}_{ij} = \langle [a_i^\#, a_j^\#]\rangle. \tag{2.30}$$

Again, with the ordering

$$\tilde{\Lambda} = \begin{pmatrix} \langle [a_k^\dagger, a_l] \rangle & \langle [a_m^\dagger, a_n^\dagger] \rangle \\ \langle [a_p, a_q] \rangle & \langle [a_u, a_v^\dagger] \rangle \end{pmatrix} \tag{2.31}$$

Theorem 2.2. *There is an affine map in between* Λ *and* $\tilde{\Lambda}$ *given by*

$$\tilde{\Lambda} = \epsilon(\Lambda) = 2\Lambda - \mathbb{1}. \tag{2.32}$$

The covariance matrix (2.22) *is then obtained as follows*

$$\frac{2}{i}\Gamma = \bar{\Omega}\tilde{\Lambda}\Omega^T = \bar{\Omega}(2\Lambda - \mathbb{1})\Omega^T, \tag{2.33}$$

where the matrix Ω *is given by* (2.10)*.*

Let us give a proof of the theorem in the simple one-mode case only. The generalization to the higher dimensions is straightforward.

Proof. Let us consider a system with one accessible mode only and denote by a, $a^\dagger$ annihilation and creation operator respectively. The anti-commutation relations then reads

$$\{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \tag{2.34}$$

We introduce Majorana hermitian operators by the relations

$$c_1 = a^\dagger + a, \quad c_2 = (-i)(a^\dagger - a). \tag{2.35}$$

The symbol for gauge-invariant state then reads $\kappa = \langle a^\dagger a \rangle$ and the generalized non-gauge-invariant one

$$\Lambda = \begin{pmatrix} \langle a^\dagger a \rangle & \langle a^\dagger a^\dagger \rangle \\ \langle aa \rangle & \langle aa^\dagger \rangle \end{pmatrix}. \tag{2.36}$$

The commutator version of this object reads

$$\tilde{\Lambda} = \begin{pmatrix} \langle [a^\dagger, a] \rangle & \langle [a^\dagger, a^\dagger] \rangle \\ \langle [a, a] \rangle & \langle [a, a^\dagger] \rangle \end{pmatrix}. \tag{2.37}$$

and is obtained by affine transformation

$$\tilde{\Lambda} = 2\Lambda - \mathbb{1} \tag{2.38}$$

This is the result of direct substitution of (2.34) into commutators

$$\begin{aligned} [a^\dagger, a] &= 2a^\dagger a - 1 \\ [a, a^\dagger] &= 2aa^\dagger - 1 \\ [a, a] &= 0 \\ [a^\dagger, a^\dagger] &= 0. \end{aligned}$$

Translation to the covariance matrix picture is then achieved by action of the map Ω, that we used in order to transform in between real and complex Hilbert space and that in this two-dimensional case reads

$$\Omega = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{2.39}$$

Let us first recall that the covariance matrix was given by the expectation values of the commutators of Majorana operators (2.35), in our case:

$$\Gamma = \frac{i}{2} \begin{pmatrix} \langle [c_1, c_1] \rangle & \langle [c_1, c_2] \rangle \\ \langle [c_2, c_1] \rangle & \langle [c_2, c_2] \rangle \end{pmatrix}. \tag{2.40}$$

The transformation $\Lambda \mapsto \Gamma$ is then given

$$\frac{2}{i}\Gamma = \bar{\Omega}\tilde{\Lambda}\Omega^T = \bar{\Omega}(2\Lambda - \mathbb{1})\Omega^T. \tag{2.41}$$

Explicitly:

$$\bar{\Omega}\tilde{\Lambda}\Omega^T = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \cdot \begin{pmatrix} \langle [a^\dagger, a] \rangle & \langle [a^\dagger, a^\dagger] \rangle \\ \langle [a, a] \rangle & \langle [a, a^\dagger] \rangle \end{pmatrix} \cdot \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} =$$

$$= \begin{pmatrix} \langle [a^\dagger, a] + [a, a] + [a^\dagger, a^\dagger] + [a, a^\dagger] \rangle & \langle i[a^\dagger, a] + i[a, a] - i[a^\dagger, a^\dagger] - i[a, a^\dagger] \rangle \\ \langle i[a^\dagger, a] - i[a, a] + i[a^\dagger, a^\dagger] - i[a, a^\dagger] \rangle & \langle [a^\dagger, a] - [a, a] + [a^\dagger, a^\dagger] - [a, a^\dagger] \rangle \end{pmatrix}$$

and comparing this with the matrix elements of (2.40)

$$\begin{aligned}
[c_1, c_1] &= [a^\dagger + a, a^\dagger + a] = [a^\dagger, a] + [a, a] + [a^\dagger, a^\dagger] + [a, a^\dagger] \\
[c_1, c_2] &= [a^\dagger + a, (-i)(a^\dagger - a)] = i[a^\dagger, a] + i[a, a] - i[a^\dagger, a^\dagger] - i[a, a^\dagger] \\
[c_2, c_1] &= [(-i)(a^\dagger - a), a^\dagger + a] = i[a^\dagger, a] - i[a, a] + i[a^\dagger, a^\dagger] - i[a, a^\dagger] \\
[c_2, c_2] &= [(-i)(a^\dagger - a), (-i)(a^\dagger - a)] = [a^\dagger, a] - [a, a] + [a^\dagger, a^\dagger] - [a, a^\dagger],
\end{aligned}$$

we see that this is exactly what was required in equation (2.41).

□

2.4.1 Example: Isotropic Gaussian states

We know that the purity of the state is equivalently expressed in terms of its covariance matrix Γ as

$$\Gamma^2 = -\mathbb{1} \tag{2.42}$$

In [11] a class of so-called isotropic Gaussian states has been introduced as follows.

Definition 2.4. *A fermionic Gaussian state is called isotropic if and only if the square of its covariance matrix Γ is invariant under any $O(2N)$ transformation, i.e.*

$$\Gamma^2 = -\lambda_0^2 \mathbb{1}, \quad \lambda_0 \leq 1. \tag{2.43}$$

As will be discussed in Chapter 6 the set of Gaussian states in not convex. Within the set described by the Definition 2.4 we can view pure states as extremal states of this class. Isotropic Gaussian states have very particular form of covariance matrix, which has crucial implication for their entanglement structure. Even though we do not study entanglement in this work, the shape of covariance matrix is still of relevance (for instance as an example of a simple environmental state, see Chapter 5).

It has been proven in [11] that the isitropic fermionic Gaussian state with the modes separated into two sectors A and B has the form (up to the local $SO(2N)$ transformation)

$$\Gamma = \Gamma_{A_1B_1} \oplus \Gamma_{A_2B_2} \oplus \cdots \oplus \Gamma_{A_SB_S} \oplus \Gamma_{A_F} \oplus \Gamma_{B_F}, \tag{2.44}$$

where $\Gamma_{A_iB_i}, i = 1, 2 \ldots S$ are of the form

$$\Gamma_{A_iB_i} = \begin{pmatrix} 0 & -\lambda_i & 0 & \kappa_i \\ \lambda_i & 0 & \kappa_i & 0 \\ 0 & -\kappa_i & 0 & -\lambda_i \\ -\kappa_i & 0 & \lambda_i & 0 \end{pmatrix}, \tag{2.45}$$

where the κ_i's and λ_i's meets the condition

$$\kappa_i^2 + \lambda_i^2 = \lambda_0^2 \tag{2.46}$$

with λ_0 from equation (2.43). $\Gamma_{A_F}, \Gamma_{B_F}$ are isotropic covariance matrices of the remaining modes (these have Williamson eigenvalue λ_0).

From an entanglement description point of view the covariance matrix form (2.44) implies the 'BCS-like' entanglement structure. This means the the entanglement of isotropic Gaussian states has a generic form of product of entangled pairs only. For the detailed description of this entanglement structure we refer to [11]. The general classification of fermionic entanglement is given in [4].

□

2.4.2 Note on the parity of Gaussian states

Let us make a short comment on the parity reduction we made in the definitions above by choosing even Gaussian states only. This is closely connected to the parity superselection rule. In all known processes in nature the parity of a closed fermionic system is always conserved. In particular, no superpositions of different parities occur. More formally, we can define the parity preserving density operators and maps as those which commute with the parity operator

$$P = \prod_i (-1)^{n_i}, \tag{2.47}$$

where by n_i we denote the occupation number of $i-$th mode.

Let us define one more notion of evenness, that is that operator as even if if has even number of Majorana generators. To conclude, from now on we will say that operator has even parity to indicate that is is parity given by the map (2.47) and that operator is even if it has even number of generators. For example, the state $|1\rangle$ is of odd parity,but the projector $|1\rangle\langle 1|$ is even operator.

Parity super selection rule implies that the physical fermionic states therefore have either even or odd number of particles. As a consequence all density operators must be even. Valid (parity preserving) unitaries are therefore generated by at least quadratic Hamiltonians. As it was already discussed every Gaussian state can be, by unitary means transformed to the standard form [62]:

$$\rho_{\mathrm{st}} = \frac{1}{2^n}\prod_{j=1}^{n}(\mathbb{1} - i\lambda_j c_{2j-1}c_{2j}). \tag{2.48}$$

Equivalently to Definition 2.1 we could define Gaussian states as those that can be by means of unitaries generated by quadratic Hamiltonians transformed into standard form (2.48).

We could, however, pick another approach, discard parity superselection rule, define Gaussian states in full generality and then impose the parity superselection condition on all operations we construct. In this Section we give this general definition of Gaussian states as those that can transformed to the standard form by means of unitaries with both quadratic and linear generators.

Throughout this work we however discuss only the states of even parity so far. Gaussian maps are then defined as those mapping Gaussian states on Gaussian states within this subspace. We can safely do that because there is a result by Knill that establishes correspondence in between the set of all Gaussian states and its even parity subspace [60].

In this Section we will use the parity unrestricted definition to outline the proof of [60] that allows us to work exclusively in the even subspace.

Unrestricted definition

Let us consider how the definition of fermionic Gaussian state would look like if we would have simply omitted the parity superselection rule.

Let us first once more recall the standard form of the Gaussian state [62]:

$$\rho_{\mathrm{st}} = \frac{1}{2^n}\prod_{j=1}^{n}(\mathbb{1} - i\lambda_j c_{2j-1}c_{2j}). \tag{2.49}$$

Then we know that any Gaussian state can be transformed into the standard form via allowed unitaries. In the even setting, unitaries are generated by Hamiltonians quadratic in the creation and annihilation operators $a^\dagger$, a. In the Majorana operators language this means that allowed Hamiltonians are of the form

$$H = \sum_{ij} K_{ij} c_i c_j, \tag{2.50}$$

where $K = K_{ij}$ are anti-hermitian complex matrix. Ignoring the parity superselection rule we can define Gaussian state as follows [14].

Definition 2.5. *State ρ is said to be generalized Gaussian if and only if it can be written in the form*

$$\rho = U\rho_{\mathrm{st}}U^{\dagger}, \tag{2.51}$$

where U is unitary operator of the form

$$U = \exp(iH_2 + iH_1), \tag{2.52}$$

where H_2 is of the form (2.50) and H_1 if the form

$$H_1 = \sum_i L_i c_i, \tag{2.53}$$

where L_i are real.

If H_1 can be chosen such as $H_1 = 0$ then we obtain even Gaussian state.

Reduction to even parity subspace

Let us now comment on the correspondence of the even parity subspace and the set of all Gaussian states. Let us define the linear map $\mathcal{E}$ that maps of $n-$qubit Gaussian states to $(n+1)-$qubit states:

$$\mathcal{E}(\rho) = V\rho \otimes \frac{\mathbb{1}}{2}V^{\dagger}, \tag{2.54}$$

where ρ denotes $n-$qubit Gaussian state and V is defined via

$$V = \exp\left(i\frac{\pi}{4}c_{2n+1}\right). \tag{2.55}$$

So what we did in (2.54) is adding $(n+1)$st maximally mixed mode and rotating the resulting state via (2.55). Because of the hermiticity of the Majorana operator c_{2n+1} we conclude that (2.55) unitary. What is then left to argue is the fact that if the state ρ is Gaussian then $\mathcal{E}(\rho)$ is Gaussian as well and find its parity.

We immediately see that when applying the map (2.54) in case that the state ρ was not even the parity is changed, which is why for the proof we need to make use of the general definition (2.51). We can then write $\mathcal{E}(\rho)$ as

$$\mathcal{E}(\rho) = (VU)\rho_{\mathrm{st}} \otimes \frac{\mathbb{1}}{2}(VU)^{\dagger}. \tag{2.56}$$

Taking into account (2.52), (2.55) and

$$\{c_{2n+1}, c_{2p}\} = 0, p = 1, \ldots, 2n. \tag{2.57}$$

We can rewrite (2.56) as

$$(VUV^\dagger)\rho_{\text{st}} \otimes \frac{\mathbb{1}}{2}(VUV^\dagger)^\dagger. \tag{2.58}$$

We need to see now what the effect of the rotation (2.55) on (2.52) is, or equivalently (given the Taylor expansion of (2.52)) to evaluate the expression

$$V(H_1 + H_2)V^\dagger. \tag{2.59}$$

It is however easy to see (given (2.57)) that

$$[V, H_2] = 0 \tag{2.60}$$

and by definition of V

$$Vc_pV^\dagger = ic_pc_{2n+1}, p = 1, \ldots, 2n. \tag{2.61}$$

So after applying (2.54) on a Gaussian state of an arbitrary parity we obtain an even Gaussian state. So we have means of mapping an arbitrary Gaussian state to an even one. As shown in [14] if $\mathcal{E}(\rho)$ is and even Gaussian the ρ is Gaussian according to Definition 2.5.

Let us conclude this section by pointing out yet another application of the generalized definition of the fermionic Gaussian states. There is a map in between fermions and qubits, so-called Jordan-Wigner transformation.Then one could study fermionic states and operations only as a subset of multi-qubit states and operations. For qubits the unitaries generated by linear Hamiltonians are of course available and the general definition of Gaussian state becomes physically relevant.

2.5 Transformations

In the Section we describe unitary transformations that preserve canonical anti-commutation relations. In other words, we want to describe which maps on the level of covariance matrices (or symbols) realize the transformation the covariance matrix of a Gaussian state into covariance matrix of another Gaussian state.

Definition 2.6. *Unitary transformation on $\mathcal{H}$ that maps every Majorana operators on $\mathcal{F}(\mathcal{H})$ onto a linear combination of Majorana operators is called Gaussian.*

In fact, in terms of covariance matrices this definition corresponds to all orthogonal transformations on the real Hilbert space [62].

Let us now provide an intuition for this statement. We know that every pure Gaussian state is a ground state of a Hamiltonian quadratic in annihilation and creation operators. In the quantum mechanical time evolution picture we therefore aim for the evolution that will lead to the ground state of another quadratic Hamiltonian. We can therefore see the transformation that preserves 'Gaussianity'

of the state as the time evolution generated by the quadratic Hamiltonians. For Majorana fermions we get

$$c_k \mapsto \sum_l O_{kl} c_l \tag{2.62}$$

in the Heisenberg picture for Majorana operators and

$$\Gamma \mapsto O\Gamma O^T \tag{2.63}$$

in Schroedinger picture for covariance matrices.

Note that while on the space $\mathcal{H}_{\mathbb{R}} = \mathrm{Re}\mathcal{H} \oplus \bar{\mathcal{H}}$ ($\mathcal{H}_{\mathbb{R}} = \mathrm{Span}\{\mathrm{Re}(x) \oplus \mathrm{Re}(\bar{y})\}$, where $x, y \in \mathcal{H}$) every orthogonal transformation maps Gaussian state on Gaussian state, not every unitary on $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \oplus \bar{\mathcal{H}}$ ($\mathcal{H}_{\mathbb{C}} = \mathrm{Span}\{x \oplus \bar{x}, x \in \mathcal{H}\}$) correspond to physical transformations on $\mathcal{B}(\mathcal{H})$. We will show below the exact structure of allowed unitaries.

Analogously to what was done in the previous section we would again like to compare and relate the maps we can apply on both covariance matrices and symbols without leaving Gaussian settings. As before we will start from gauge-invariant settings introduce in [31], then we will generalize to non-gauge invariant situation and finally compare with transformation of covariance matrices.

Let us start from gauge-invariant maps and the CAR-algebra built on Hilbert space $\mathcal{H}$. It is straightforward to define unitary, which preserves CAR relations:

$$a_k \mapsto \sum_l U_{kl} a_l \tag{2.64}$$

in Heisenberg picture for annihilation operators, where the same can of course be done for creation operators. But we see that by unitaries we can only achieve annihilation operator to be mapped on the linear combination of annihilation operators and creation on creation operators, which is why that under such transformation we stay in the gauge-invariant setting.

Looking at (2.62) that is certainly not true in the case of the transformation of Majorana modes. So we need to introduce Bogoliubov transformations, well known from the condensed matter physics. Bogoliubov transformation in general map the annihilation operator on the linear combination of both creation and annihilation operators. So in out case this is exactly the way how to leave gauge-invariant setting while staying in the Gaussian now provided that the resulting transformation will be unitary.

Let us define

$$a_k \mapsto \sum_l K_{kl} a_l + L_{kl} a_l^\dagger. \tag{2.65}$$

Note that annihilation operator a is defined to be complex anti-linear and creation operator $a^\dagger$ complex linear map on the underlying Hilbert space (meaning $\phi \mapsto a(\phi)$ is complex anti-linear $\forall \phi \in \mathcal{H}$ and $\phi \mapsto a^\dagger(\phi)$ is complex linear $\forall \phi \in \mathcal{H}$).

Since we fixed the basis of the Hilbert space $\mathcal{H}$ we can work directly with the linear combinations of the creation and annihilation operators. Nonetheless we have to careful to preserve correspondence in between transformations on the one-particle Hilbert space and transformations in the Fock space. Without the notation (2.6) the equation (2.65) would read

$$a(\phi) \mapsto a(K\phi) + a^\dagger(L(\phi)), \quad \forall \phi \in \mathcal{H}, \tag{2.66}$$

which is why we have to conclude that while K will be linear transformation, L must be anti-linear. Imposing the requirement that canonical anti-commutation relations must be preserved we obtain following additional constraints on K and L:

$$K^*L + L^*K = 0 \qquad \text{and} \qquad K^*K + L^*L = 1. \tag{2.67}$$

These can be easily checked by direct computation of canonical anticommutation relations for Bogoliubov-transformed operators.

Instead of working with both linear and anti-linear operators on $\mathcal{H}$ we choose to enlarge the underlying Hilbert space into $\mathcal{H}_\mathbb{C}$, where we can define general transformation as [29]

$$R = \begin{pmatrix} Q & P \\ \bar{P} & \bar{Q} \end{pmatrix} \tag{2.68}$$

where P, Q are both linear mappings acting on

$$Q : \mathcal{H} \to \mathcal{H} \tag{2.69}$$

$$P : \bar{\mathcal{H}} \to \mathcal{H}. \tag{2.70}$$

Now all operators are real linear. The additional constraints (2.67) now translate into

$$\begin{aligned} P^*P + Q^T\bar{Q} &= \mathbb{1}, \\ P^T\bar{Q} + Q^*P &= 0, \\ PP^* + QQ^* &= \mathbb{1}, \\ PQ^T + QP^T &= 0. \end{aligned} \tag{2.71}$$

Note that onthe space $\mathcal{H}_C$ the gauge-invariant map looks like

$$R = \begin{pmatrix} Q & 0 \\ 0 & \bar{Q} \end{pmatrix}. \tag{2.72}$$

Enlarging the Hilbert space in order to have the freedom to define both gauge invariant and non-invariant transformation we now work in the same dimension as we needed to construct the covariance matrix (2.22). The only thing left to do is thus to find proper map which would translate allowed unitaries on complex Hilbert space into orthogonal transformation onto real one. But we already did that by constructing (2.41) mapping generalized symbol on covariance matrix and vice versa.

$$O = \frac{1}{2}\bar{\Omega}R\Omega^T, \tag{2.73}$$

where Ω is (up to normalization) an orthogonal transformation on $\mathcal{H}_{\mathbb{R}}$.

Let us conclude by writing explicitly the action of (2.41) on the gauge-invariant symbol $\begin{pmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{pmatrix}$:

$$\begin{pmatrix} Q & P \\ \bar{P} & \bar{Q} \end{pmatrix} \cdot \begin{pmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{pmatrix} \cdot \begin{pmatrix} \bar{Q} & P \\ \bar{P} & Q \end{pmatrix} = \begin{pmatrix} Q\Lambda\bar{Q} + P\bar{\Lambda}\bar{P} & Q\Lambda P + P\bar{\Lambda}Q \\ \bar{P}\Lambda\bar{Q} + \bar{Q}\bar{\Lambda}\bar{P} & \bar{Q}\bar{\Lambda}Q + \bar{P}\Lambda P \end{pmatrix}. \tag{2.74}$$

In this section the equivalence of covariance matrix and generalized symbol was established as well as the form of the general quasifree transformation of both symbol and covariance matrix. Physically speaking the occurrence of the nonzero expectation values of the terms $a^\dagger a^\dagger$ and aa means that the creation or annihilation of pairs of particles is allowed and, in particular, superpositions of states with different particle number. This means that the parity of the system is undisturbed, but particle number conservation is not obeyed. This is achieved via interaction with some environmental state (note that in our setting all gauge-invariant operations are in the commutant of number operator $N = a^\dagger a$).

Chapter 3

Fermionic Gaussian States: Grassmann Approach

3.1 Motivation

Traditional language of quantum optics are the so-called coherent states, that are left-eigenstates of the bosonic annihilation operators. Their intrinsic property is that particle number of such state is indefinite, while the phase is determined exactly. In this manner they can be seen as a counterpart of the Fock states with fully determined particle number and indefinite phase. In quantum optics, coherent states turned out to be proper basis for describing all states of electromagnetic field (see for instance [71]).

Bosonic fields are then described as a superposition of coherent states and the associated weight functions can be regarded as quasi-probability distributions in phase space. There is quite a variety of function that play an important role in describing density operators.

It is a natural question to ask whether we can do the same in the case of fermion fields (see [19]). Let us try to introduce coherent state in an exactly analogous way as it has done for bosons, as an eigenstate of an annihilation operator. Let a, $a^\dagger$ be fermionic annihilation and creation operator respectively. We define an eigenstate of a as

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{3.1}$$

Let us now take two different annihilation operators a_i, a_j, $i \neq j$. In that case we obtain eigenequations

$$a_i\,|\alpha\rangle = \alpha_i\,|\alpha\rangle \tag{3.2}$$
$$a_j\,|\alpha\rangle = \alpha_j\,|\alpha\rangle \tag{3.3}$$

and the anticommutation relation $\{a_i, a_j\} = 0$. Then making use of (3.1) we can write either

$$a_i a_j\,|\alpha\rangle = a_i \alpha_j\,|\alpha\rangle = \alpha_i \alpha_j\,|\alpha\rangle \tag{3.4}$$

or

$$a_i a_j \left|\alpha\right\rangle = -a_j a_i \left|\alpha\right\rangle = -\alpha_i \alpha_j \left|\alpha\right\rangle . \tag{3.5}$$

From these equations it is obvious that RHS of (3.4) must be equal to RHS of (3.5). For complex numbers α_i, α_j this can only be true if at least one of the eigenvalues is zero. Thus the fermionic vacuum is then the only physically admissible coherent state. One possibility how not leave the coherent-like states framework is to say that not only fermionic operators anti-commute but that their eigenvalues are anti-commuting objects as well [74]. This means that within the rules of Grassmann algebra (see [10]) we can at least formally introduce coherent states and recover the quasi-probability distributions and correlation functions analogously to bosonic fields [19]. Grassmann algebra tools can be used not only to describe Gaussian states but also Gaussian maps [15].

In spite of the fact that when deriving our results we did not make use of the Grassmann algebra, we include this chapter in order to give to a complete picture of the methods currently used in fermionic Gaussian approximation. The Grassamann algenra tools alone go beyond the Gaussian approximation, the can be, however, used to prove some important statements in particularly elegant form. In this chapter we will mostly follow [19],[15] and the papers cited thereby. The derivation of the results quoted below can be found there in more detailed manner.

3.2 Grassmann variables

Let us consider a fermionic system described by creation and annihilation operators $a_i^\dagger$, a_j and recall once more their anticommutation relations

$$\{a_i^\dagger, a_j\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \tag{3.6}$$

Grassmann numbers will be denoted by γ_i and their complex conjugates by γ_i^*. They satisfy the relations

$$\{\gamma_i, \gamma_j\} = \{\gamma_i^*, \gamma_j\} = \{\gamma_i^*, \gamma_j^*\} = 0, \quad \forall i, j. \tag{3.7}$$

In addition to that it is assumed that Grassmann number anticommute with fermionic operators:

$$\{\gamma_i, a_j\} = 0. \tag{3.8}$$

Analogously to the definition for operators the order of operation will be reversed under hermitian conjugation for both Grassmann numbers and fermionic operators.

Let us denote that while all the operators and Grassmann number anti-commute with each other, when annihilation operator is multiplied by Grassmann number their anti-commutativity properties cancel . It can be easily checked that

$$[a_i^\dagger \gamma_i, \gamma_i^* a_i] = 0, \quad \text{for} \quad i \neq j. \tag{3.9}$$

3.3 Fermionic coherent states

Based on the discussion in the Section 3.1 we will proceed in defining the fermionic coherent framework in exact analogy with the bosonic coherent state with arguments being Grasmann numbers instead of complex numbers.

As the bosonic coherent states can be seen as the displaced vacuum states in the phase space we start by constructig displacement operator for Grassmann variables and this will be later use to introduce fermionic coherent state.

Definition 3.1. *Displacement operator $D(\gamma)$ for $\gamma = (\gamma_1, \ldots, \gamma_n)$ being vector of Grassmann numbers is defined as*

$$D(\gamma) = \exp\left(\sum_i \left(a_i^\dagger \gamma_i - \gamma_i^* a_i\right)\right). \tag{3.10}$$

Using (3.9) the expression (3.10) can be rewritten as

$$D(\gamma) = \prod_i \exp\left(a_i^\dagger \gamma_i - \gamma_i^* a_i\right) = \prod_i \left(1 + a_i^\dagger \gamma_i - \gamma_i^* a_i + \left(a_i^\dagger a_i + \frac{1}{2}\right)\gamma_i^* \gamma_i\right). \tag{3.11}$$

Then it is easy to check that the displacement operator is unitary. Using (3.11) it is then straighforward to see that

$$D^\dagger(\gamma) a_i D(\gamma) = a_i + \gamma_i \tag{3.12}$$

and

$$D^\dagger(\gamma) a_i^\dagger D(\gamma) = a_i^\dagger + \gamma_i^*. \tag{3.13}$$

Definition 3.2. *A fermionic coherent state $|\gamma\rangle$ is then defined as vacuum displaced by* (3.10)

$$|\gamma\rangle = D(\gamma)\,|0\rangle\,. \tag{3.14}$$

Let us now check that $|\gamma\rangle$ is indeed an eigenstate of every annihilation operator:

$$\begin{aligned} a_i\,|\gamma\rangle &= a_i D(\gamma)\,|0\rangle = D(\gamma)D^\dagger(\gamma)a_i D(\gamma)\,|0\rangle = \\ &= D(\gamma)(a_i + \gamma_i)\,|0\rangle = D(\gamma)\gamma_i\,|0\rangle = \gamma_i D(\gamma)\,|0\rangle = \\ &= \gamma_i\,|\gamma\rangle\,, \quad \forall i. \end{aligned} \tag{3.15}$$

What is an interesting difference from bosons is the fact that, thanks to Pauli exclusion principle, we can also define an eigenstate of the creation operator for fermions (only in this abstract Grassmann settings though). Let us consider the state of the system under consideration in which every mode is filled:

$$|1\rangle = \prod_i a_i^\dagger\,|0\rangle\,. \tag{3.16}$$

Definition 3.3. *The common eigenstate of all fermionic creation operators is the fully filled Fermi sea displaced by D:*

$$|\tilde{\gamma}\rangle = D(\gamma)\,|1\rangle \tag{3.17}$$

It can be checked (same calculation as (3.15)) that $|\tilde{\gamma}\rangle$ is an eigenstate of all fermionic creation operators $a_i^\dagger$. I.e. while the eigenstate of annihilation operators can be seen as dispaced vacuum state, the eigenstate of creation operators can be viewed as displaced maximally occupied state.

The states $|\gamma\rangle$ and $|\tilde{\gamma}\rangle$ described above are just two particular,yet examples of fermionic Gaussian states. They were constructed using particular choice of operator (3.10). We will show how to construct wider class of Gaussian operators that maps the set of Gaussian states on itself. Before giving the general characterization of Gaussian states and maps in terms of Grassmann variables we however need to review some basic results of Grassmann calculus.

3.4 Brief review of Grassmann calculus

In this section we will introduce the integration and differentiation with respect to Grassmann variables to set the basis for the goal of rewriting the formulas for Gaussian integrals within the Grassmann framework.

Firstly let us note that the direct result of the relations (3.7) is that for any Grassmann number γ we have

$$\gamma^2 = 0. \tag{3.18}$$

The most general function of one Grassmann variable is therefore the linear one, i.e.

$$f(\gamma) = a\gamma + b, \tag{3.19}$$

where a, b are complex numbers.

Having functions we can define the differentiation and integration with respect to Grassmann variables.

Definition 3.4. *Differentiation with respect to Grassmann variable γ_j is defined via the relations*

$$\frac{\partial \gamma_i}{\partial \gamma_j} = \delta_{ij}, \quad \frac{\partial}{\partial \gamma_i} 1 = 0 \tag{3.20}$$

and the Leibniz rule

$$\frac{\partial}{\partial \gamma_i}(\gamma_j f(\gamma)) = \delta_{ij} f(\gamma) - \gamma_j \frac{\partial}{\partial \gamma_i} f(\gamma). \tag{3.21}$$

The equality (3.21) enforces an anticommutation of partial derivatives:

$$\left\{ \frac{\partial}{\partial \gamma_i}, \frac{\partial}{\partial \gamma_j} \right\} = 0. \tag{3.22}$$

Definition 3.5. *Integration with respect Grassmann variable is defined [10]*

$$\int d\gamma = 0, \quad \int d\gamma_i \gamma_j = \delta_{ij}. \tag{3.23}$$

Applying this definition to general function (3.19) we obtain

$$\int d\gamma f(\gamma) = \int d\gamma (a\gamma + b) = a. \tag{3.24}$$

On the other hand

$$\frac{\partial}{\partial \gamma} f(\gamma) = \frac{\partial}{\partial \gamma}(a\gamma + b) = a. \tag{3.25}$$

So we see that performing the integration over a Grassmann variable is the same operation like differentiation with respect to it.

Since the main purpose of this chapter is to give the expressions for Gaussian integrals the next step is to use above definitions to evaluate the integral of an exponential function. Let us consider two different Grassmann numbers $\gamma_i \neq \gamma_j$ and a complex number c. Then having in mind the property (3.18) and the definition (3.23) we can compute

$$\begin{aligned} \int d\gamma_i d\gamma_j \exp(-c\gamma_i\gamma_j) &= \int d\gamma_i d\gamma_j (1 - c\gamma_i\gamma_j) = \\ &= -c \int d\gamma_i d\gamma_j \gamma_i\gamma_j = c \int d\gamma_i\gamma_i d\gamma_j\gamma_j = c. \end{aligned} \tag{3.26}$$

Now we have all the necessary ingredients to evaluate Gaussian integrals. We only list the results necessary for the following chapters. The calculation is rather straightforward using the rules listed above. For the details we refer to [24].

Theorem 3.1. *For a vectors of Grassmann numbers γ, η and complex anti-symmetric matrix M we have*

$$\int D\gamma \exp\left(\frac{i}{2}\gamma^T M \gamma\right) = i^n \mathrm{Pf}(M) \tag{3.27}$$

and

$$\int D\gamma \exp\left(\frac{i}{2}\gamma^T M \gamma + \eta^T \gamma\right) = i^n \mathrm{Pf}(M) \exp\left(-\frac{i}{2}\eta^T M^{-1} \eta\right). \tag{3.28}$$

3.5 Gaussian states and operators

In this section we will introduce the Grassmann version of the covariance matrix approach described in Section 5.4 introduced in [15]. The connection in between two approaches is the map assigning the Grassmann variable to each Majorana operator.

Definition 3.6. *Consider X to be a linear operator on the Clifford algebra $\mathcal{C}_{2n}$ that is generated by Majorana operators $c_{2j-1} = a^\dagger + a$, $c_{2j} = (-i)(a_j^\dagger - a_j)$ that fulfill the anti-commutation relation $\{c_a, c_b\} = 2\delta_{ab}$, $\forall 1 \leq a, b \leq 2n$. To every such map can be assigned polynomial in Grassmann variables via*

$$\omega(c_p, c_q, \dots c_r, \gamma) = \gamma_p \gamma_q \dots \gamma_r, \quad \omega(\mathbb{1}, \gamma) = 1, \tag{3.29}$$

where $\gamma_j \in \mathcal{G}_{2n}$, the algebra of n Grassmann variables.

Then we define a state to be Gaussian in terms of Gaussianity of its Grassmann representation. The state ρ of n fermionic modes on $\mathcal{C}_{2n}$ is then defined to be Gaussian if its Grassmann representation is Gaussian, i.e.

$$\omega(\rho, \gamma) = \frac{1}{2^n} \exp\left(\frac{i}{2}\gamma^T \Gamma \gamma\right), \tag{3.30}$$

where Γ is $2n \times 2n$ real antisymmetric matrix.

Bravyi calls Γ the correlation matrix of the state and defines it via

$$\Gamma_{ij} = \frac{i}{2}\mathrm{Tr}(\rho[c_i, c_j]), \tag{3.31}$$

which exactly corresponds to the definition of covariance matrix (see Eq. (2.22)) from [62].

Definition 3.7. *An operator X on the Clifford algebra $\mathcal{C}_{2n}$ with $\mathrm{Tr}X \neq 0$ is called even Gaussian if its Grassmann representation is*

$$X(\gamma) = C \exp\left(\frac{i}{2}\gamma^T \Gamma \gamma\right), \tag{3.32}$$

where C is complex number and Γ is $2n \times 2n$ complex antisymmetric matrix, the correlation matrix of X. In case that $\mathrm{Tr}X = 0$, we say that X is Gaussian if there is a sequence $\{X_m\} \in \mathcal{C}_{2n}$ of Gaussian operators with $\mathrm{Tr}X_m \neq 0, \forall m$ such that $X = \lim_{m\to\infty} X_m$.

There is operational necessary and sufficient criterion for operator X to be Gaussian [15]:

Theorem 3.2. *An operator $X \in \mathcal{C}_{2n}$ is Gaussian iff X is even and satisfies*

$$[\hat{\Lambda}, X \otimes X] = 0, \quad \hat{\Lambda} = \sum_i^{2n} c_i \otimes c_i. \tag{3.33}$$

This statement gives us direct way of checking Gaussianity. We will make use of this Theorem in Chapter 6.

3.6 Gaussian Linear Maps

We now consider linear map on $\mathcal{C}_{2n}$. Such maps can be defined via their Grassmann representation.

Definition 3.8. *A linear map $\varepsilon : \mathcal{C}_{2n} \to \mathcal{C}_{2n}$ is Gaussian if and only if it admits representation*

$$\varepsilon(X)(\gamma) = C \int \exp[S(\gamma, \eta) + i\eta^T \mu] X(\mu) D\eta D\mu, \tag{3.34}$$

where

$$S(\gamma, \eta) = \frac{i}{2}(\gamma^T, \eta^T) \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} \begin{pmatrix} \gamma \\ \eta \end{pmatrix} \tag{3.35}$$

for some $2n \times 2n$ complex matrices A, B, D and some complex number C. The function $S(\gamma, \eta)$ is called an action of the map ε. Looking at structure of (3.35) we see that we can assume both A and D to be anti-symmetric (relevant terms have the form $\gamma^T A\theta$ and $\eta^T D\eta$).

Equations (3.34) and (3.35) together with the identities (3.27), (3.28) allow us easily derive how the transformation by means of Gaussian linear map looks like on the covariance matrix level:

$$\varepsilon(\Gamma) = B(\mathbb{1} + \Gamma D)^{-1} \Gamma B^T + A. \tag{3.36}$$

Since we are aiming to find a classification of Gaussian channels we would like to restrict the maps described by (3.36) to those that are completely positive and trace preserving. As shown in [15] the requirement on ε to be trace-preserving reduces the expression (3.36) to the simple formula

$$\varepsilon(\Gamma) = B\Gamma B^T + A. \tag{3.37}$$

Complete positivity is then solved by imposing restriction on the matrix in the functional representation of the map ε (3.35), in particular the following statement holds

Theorem 3.3. *The map ε as in Definition 3.8 is completely positive if and only if the constant C is positive and the matrix*

$$M = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} \tag{3.38}$$

is real and meets the condition

$$M^T M \leq \mathbb{1}. \tag{3.39}$$

Let us conclude this Chapter by illustrating rather abstract framework introduced above on an easy example [15].

3.6.1 Example: Grassmann representation of the projector

working in the state of n fermionic modes, we eill consider the projector on the state 'the first mode is empty', i.e. $a_1 a_1^\dagger$. In order to make use of (3.29) we need to express $a_1 a_1^\dagger$ in terms of Majorana operators:

$$a_1 a_1^\dagger = \frac{1}{2}\left(\mathbb{1} + i c_1 c_2\right). \tag{3.40}$$

Applying (3.29) we obtain

$$\omega(a_1 a_1^\dagger, \gamma) = \frac{1}{2}\left(\mathbb{1} + i c_1 c_2\right) = \frac{1}{2}\exp(i\gamma_1\gamma_2). \tag{3.41}$$

The trace of the product of XY, where $X, Y \in \mathcal{C}_{2n}$ can be computed as [15]

$$\mathrm{Tr}(XY) = \frac{1}{2^n}\int D\gamma D\mu \exp(\gamma^T \mu)\omega(X,\gamma)\omega(Y,\mu), \tag{3.42}$$

which applied in our case reads

$$\mathrm{Tr}(\rho a_1 a_1^\dagger) = \frac{1}{2^n}\int D\gamma D\mu \exp(\gamma^T \mu)\omega(a_1 a_1^\dagger,\gamma)\omega(\rho,\mu). \tag{3.43}$$

Let us introduce K, $2n \times 2n$ antisymmetric matrix with only two non-zero entries: $K_{12} = 1$ and $K_{21} = -1$, so that we can write $\omega(a_1 a_1^\dagger, \gamma) = \frac{1}{2}\exp\left(\frac{i}{2}\gamma^T K \gamma\right)$. Therefore we can write

$$\mathrm{Tr}(\rho a_1 a_1^\dagger) = \frac{(-1)^n}{2}\int D\gamma \exp\left(\frac{i}{2}\gamma^T K\gamma\right)\int D\mu \exp\left(\gamma^T\mu + \frac{i}{2}\mu^T M \mu\right) = \tag{3.44}$$

$$= \frac{(-1)^n}{2}\mathrm{Pf}(M)\int D\gamma \exp\left(\frac{i}{2}\gamma^T(K - M^{-1})\gamma\right) = \tag{3.45}$$

$$= \frac{1}{2}\mathrm{Pf}(M)\mathrm{Pf}(K - M^{-1}). \tag{3.46}$$

Here we made use if identities (3.27) and(3.28) If the matrix Γ is be singular the resulting expression may be regularized using $\mathrm{Pf}(L)^2 = \mathrm{Det}(L)$ (this holds for every anti-symmetric matrix):

$$\left[\mathrm{Tr}(\rho a_1 a_1^\dagger)\right]^2 = \frac{1}{4}\mathrm{Det}(MK - 1), \tag{3.47}$$

where we used the multiplicativity of the determinant.

□

Chapter 4

Extremality of Gaussian States

4.1 Introduction

In this Chapter we explore extremality properties of fermionic Gaussian states. It is a well known fact that Gaussian states maximize the von Neumann entropy at a given energy (see cf. [50]).

This statement was a motivation for extremality considerations within a more general framework of a wider class of functionals with physically relevant properties. Extremality of bosonic Gaussian states with respect to the certain class of functionals has been proven in [87]. This result leads to lower bounds on entanglement-like quantities and upper bounds on quantum channels capacities.

The proof in [87] relies on the equivalence of pointwise convergence of characteristic functions and strong convergence of density operators [28] and the non-commutative central limit theorem [26].

We would like to check whether this extremality argument is also valid for fermionic Gaussian states. In the previous Chapter we learned that exploring phase space convergence for fermions would require Grassmann algebra techniques. There is, however, one difference between the fermionic and the bosonic setting and that is the dimensionality of the underlying Hilbert space. While in the bosonic case the Hilbert space is of infinite dimension, in the fermionic case it is finite dimensional. For fermions, we are therefore able to explore the convergence directly on the level of density matrices.

Our extremality proof is very much analogous to the approach of [87] up to the point that we are able to avoid to the phase space treatment. For a limiting argument we made use of the central limit theorem for anti-commuting operators proven in [51].

In this Chapter, we give the extremality proof in terms of operators on Fock space since this is the language chosen to present all our results throughout this work. We argue, however, that this proof can be generalized to an arbitrary representation of canonical anti-commutation relations.

4.2 The proof

The statement of the theorem we intend to prove is following [87].

Theorem 4.1. *Let $f : \mathcal{H}^{\otimes d}) \to \mathbb{R}$ be a continuous functional, which is strongly super-additive and invariant under local unitaries $[f(U^{\otimes d}\rho U^{\dagger\otimes d}) = f(\rho)]$.*
Then for every even density operator ρ describing d-partite fermionic system with finite first and second moments we have that $f(\rho) \geq f(\rho_G)$, where ρ_G is the even fermionic Gaussian state with the same 2nd moments as ρ.

Let us give the precise definition of the notion of strong super-additivity.

Definition 4.1. *Let ρ be a density operator on $\mathcal{H} := (\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) \otimes (\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2})$ and ρ_i, $i = 1, 2$ restrictions of ρ on $\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$. Then the functional $f : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ is called super-additive if and only for all ρ it holds*

$$f(\rho) \geq f(\rho_1) + f(\rho_2) \tag{4.1}$$

(with natural generalization to bigger number of parties).

Analogously we can define so-called strong sub-additivity (see Definition 5.11). We will make use of this when applying this theorem in the context of quantum channels (Chapter 5).

The basic idea of the proof, as in bosonic case, is the following:

$$f(\rho) = \frac{1}{n} f(\rho^{\otimes n}) = \frac{1}{n} f(\underbrace{U^{\otimes d}\rho^{\otimes n}U^{\dagger\otimes d}}_{\tilde{\rho}}) \tag{4.2}$$

$$\geq \frac{1}{n}\sum_{k=1}^{n} f(\tilde{\rho}_k) \to f(\rho_G), \tag{4.3}$$

where the additivity, invariance under local unitaries and strong super-additivity respectively is used. The last step will be in our case the result of the application of the quantum mechanical central limit theorem for anti-commuting variables [51].

Before proving the theorem we need to introduce the notation. Let us now consider a d-dimensional fermionic system and denote the associated creation and annihilation operators as $a_i^\dagger$, a_i respectively, where $i = 1, \ldots d$. Let us take n copies of this system. The creation and annihilation operators for the $i-$th mode of the $j-$th copy will be denoted $a_i^{j\dagger}$, a_i^j respectively, where $j = 1, \ldots n$. By $a_i^{j\dagger}$ we denoted $(a_i^j)^\dagger$. Taking these copies corresponds to the first equality in the equation (4.2), for the second one we want to choose an appropriate local unitary. This desirable transformation, again as in the case of bosons, is the Hadamard transformation (see [68]). Then, in order to take the limit as required in the equation (4.3), we send the number of copies of the system to infinity.

Let us start by noting the independence of the creation and annihilation operators for the different copies of the system. Indeed for $j \neq k$ it holds

$$\{a_i^j, a_{i'}^k\} = \{a_i^{j\dagger}, a_{i'}^{k\dagger}\} = 0 \tag{4.4}$$

$$\{a_i^j, a_{i'}^{k\dagger}\} = 0, \quad i, i' = 1, \ldots, d. \tag{4.5}$$

From now on we will use the following notation

$$a^j := \{a_i^j\}_{i=1}^d. \tag{4.6}$$

We can see each of these sets as a realization of CAR over $\mathbb{C}^d$. It is clear that the observables built from a^j are (for different j) are both independent (by (4.4)) and identically distributed in the state $\rho^{\otimes n}$ (this is true by construction, we take n copies of the same system).

Let us start by making the convergence argument of equation (4.3).

4.2.1 Central limit theorem

In order to make the convergence argument we need to introduce so-called cumulant.

Definition 4.2. *Let $a = \{a_i\}_{i=1}^d$ be the set of annihilation operators for the $d-$mode fermionic system in the even state ρ. Setting $a_i^{\#}$ to be a_i if $\# = 0$ and $a_i^\dagger$ if $\# = 1$ for all i we define the cumulant of the $m-$th order as*

$$K_m = \mathrm{Tr}(\rho a_1^{\#_1} \dots a_m^{\#_m}) = \tag{4.7}$$

$$= \sum_P \delta_P K_{p_1}(n_{11}, \#_{11}, \dots n_{1p_1}, \#_{n_{1p_1}}) \dots K_{p_r}(n_{r1}, \#_{n1}, \dots n_{rp_r}, \#_{n_{rp_r}}), \tag{4.8}$$

where the sum is over all partitions

$$P = \{(n_{11}, \dots, n_{1p_1}), \dots, (n_{r1}, \dots, n_{rp_r})\} \tag{4.9}$$

of $M = \{1, \dots m\}$ into r disjoint subsets of even cardinality p_l, where $l = 1, \dots r$; δ_P is the sign of the permutation.[1] *The indices in the argument of the cumulant denotes the indices of creation resp. annihilation operators to be taken into account in the sum.*

Let us explain the notation further in the following example.

4.2.2 Example: 2nd order cumulants

Let us consider the cumulants of the second order only. Note that in our physical approach of taking copies this means that we take two copies of the system and therefore we have available two sets of annihilation operators a^1 and a^2 describing the overall system.

First of all, let us write the cumulant of second order for the set of operators a^1. Since we are going to work with cumulants for different copies of our system, we will denote the relevant set of operators by superscript.

[1] Under permutation we understand the map of $M = \{1, \dots m\}$ onto $P = \{(n_{11}, \dots, n_{1p_1}), \dots, (n_{r1}, \dots, n_{rp_r})\}$; $\delta_P = 1$ if the elements of M have been exchanged an even number of times, $\delta_P = -1$ otherwise.

$$K_2^{a^1}(1,\#_1,2,\#_2) = \mathrm{Tr}(\rho(a_1^{1\#_1}a_2^{1\#_2})) \tag{4.10}$$

Let us therefore continue by writing the cumulant for the set $\frac{1}{\sqrt{2}}(a^1+a^2)$ and, for further convenience also for the linear combination $\frac{1}{\sqrt{2}}(a^1-a^2)$.

$$\begin{aligned}
&K_2^{-\frac{1}{\sqrt{2}}(a^1+a^2)}(1,\#_1,2,\#_2) = \\
&= \frac{1}{2}\mathrm{Tr}(\rho(a_1^{1\#_1}+a_1^{2\#_1})(a_2^{1\#_2}+a_2^{2\#_2})) = \\
&= \frac{1}{2}(K_2^{a^1}(1,\#_1,2,\#_2)+K_2^{a^2}(1,\#_1,2,\#_2))+ \\
&+ \frac{1}{2}\mathrm{Tr}(\rho(a_1^{1\#_1}a_2^{2\#_2}+a_1^{2\#_1}a_2^{1\#_2})) = \\
&= \frac{1}{2}(K_2^{a^1}(1,\#_1,2,\#_2)+K_2^{a^2}(1,\#_1,2,\#_2)),
\end{aligned}$$

where the term $\frac{1}{2}\mathrm{Tr}(\rho(a_1^{\#_1 1}a_2^{\#_2 2}+a_1^{\#_1 2}a_2^{\#_2 1}))$ vanishes because of eveness of ρ and independence of the copies specified by a^1, a^2. For the second linear combination we obtain

$$\begin{aligned}
&K_2^{-\frac{1}{\sqrt{2}}(a^1-a^2)}(1,\#_1,2,\#_2) = \\
&= \frac{1}{2}\mathrm{Tr}(\rho(a_1^{1\#_1}+a_1^{2\#_1})(a_2^{1\#_2}+a_2^{2\#_2})) = \\
&= \frac{1}{2}(K_2^{a^1}(1,\#_1,2,\#_2)+K_2^{a^2}(1,\#_1,2,\#_2))+ \\
&- \frac{1}{2}\mathrm{Tr}(\rho(a_1^{1\#_1}a_2^{2\#_2}+a_1^{2\#_1}a_2^{1\#_2})) = \\
&= \frac{1}{2}(K_2^{a^1}(1,\#_1,2,\#_2)+K_2^{a^2}(1,\#_1,2,\#_2))
\end{aligned}$$

□

Results for second order cumulants can be readily extended to the cumulants of the general order. This claim is contained in the following Theorem and Corollary proven in [51].

Theorem 4.2. *Let $a^1, a^2, \ldots, a^n$ be sets of annihilation operators for n-copies of the system under consideration. Then*

$$K_m^{\frac{1}{\sqrt{n}}(a^1+a^2+\cdots+a^n)} = n^{-\frac{1}{2}m}\sum_{j=1}^{n} K_m^{a^j} \tag{4.11}$$
$$\text{where} \quad m = 2,4,\ldots.$$

Corollary 4.1. *Let $a^1, a^2, \ldots, a^n$ be sets of annihilation operators for n-copies of the system under consideration. Then*

$$K_m^{\frac{1}{\sqrt{n}}(a^1+a^2+\cdots+a^n)} = n^{1-\frac{1}{2}m}K_m^{a^1}. \tag{4.12}$$

Having the result of the Corollary (4.1) it is easy to compute the limit from the equation (4.3).

Corollary 4.2.

$$\lim_{n\to\infty} K_m^{\frac{1}{\sqrt{n}}(a_1+a_2+\cdots+a_n)} = K_2^{a_1} \quad \text{for} \quad m=2 \tag{4.13}$$

and vanishes otherwise (i.e. for $m > 2$).

We define the convergence for the density operators on the Fock space as follows

Definition 4.3. *We say that a sequence a^n, $n = 1, 2, \ldots$ with specified state ρ_{a^n} converges in distribution to the limit state ρ_0 if for A, an arbitrary operator on Fock space*

$$\lim_{n\to\infty} \mathrm{Tr}(\rho_{a^n} A) = \mathrm{Tr}(\rho_0 A). \tag{4.14}$$

Using finite dimensionality of Fock space we can conclude its compactness [70] which implies that every bounded sequence of operators on this space has convergent subsequence. If we choose this bounded sequence to be

$$\rho_n = \rho_{\frac{1}{\sqrt{n}}(a^1+a^2+\cdots+a^n)} \tag{4.15}$$

we can conclude that the sequence ρ_n has subsequence $\{\rho_{n^j}\}$ that converge to some limiting distribution ρ_0. Thus we get (according to our previous definition of convergence 'in distribution') for arbitrary operator A on Fock space:

$$\lim_{j\to\infty} \mathrm{tr}(\rho_{n^j} A) = \mathrm{tr}(\rho_0 A). \tag{4.16}$$

If we denote the set of annihilation operators specifying the limiting state ρ_0 by a^0 and take into account the definition of the cumulant (Definition 4.2) we can conclude that cumulants of a^0 in the state ρ_0 are the limits of the cumulants of a^0 in the states ρ_{n_j} (limits of $K_m^{\frac{1}{\sqrt{n_j}}(a^1+a^2+\cdots+a^{n_j})}$, $m = 2, 4, \ldots$). But we know from Corollary 4.2 that these all vanish unless $m = 2$. Thus the liming state is Gaussian.

Let us elaborate further on the meaning of the equation (4.16). We see that for the sequence of cumulants of an arbitrary even state ρ over any combination of a^i's there exists a set of annihilation operators a^0 and corresponding state ρ_0 for which the sequence of cumulants vanish except of K_2. Indeed the state fully specified by second order cumulants only exactly correspond to our definition of Gaussian states from Chapter 2. Afterall, the definition of cumulant agrees exactly with the statement of Theorem 2.1.

So far we proved that having the sequence of $a^1, a^2, \ldots$, which are independent and identically distributed in the state ρ implies that

$$\lim_{n\to\infty} \mathrm{Tr}(\rho_{\frac{1}{\sqrt{n}}(a^1+a^2+\cdots+a^n)} A = \mathrm{Tr}(\rho A), \tag{4.17}$$

where ρ is Gaussian state. This corresponds to the central limit theorem as stated in [51].

4.2.3 The extremality argument

Let us now put together all the arguments suggested above so that we can conclude the proof.

Let us have $d-$partite system in the state ρ. Next we take n copies of this system. Each of these is characterized by the set of operators of a^j, $j = 1, \dots n$. Let us perform local Hadamard transformation of this enlarged system. By locality we mean that the overall unitary operator has the structure of the tensor product of d Hadamard transformations acting on each of the parties. In particular,

$$H = \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right]^{\otimes l} . \tag{4.18}$$

For convenience, we chose $n = 2^l$ and l even.

By means of local Hadamard transformations we can reach the situation where the set of annihilation operators associated to the modes of the first system a^1 will be transformed into $\frac{1}{\sqrt{n}}(a^1 + a^2 + \dots + a^n)$. For the other copies we obtain similar linear combinations with the difference that $\frac{n}{2}$ of the coefficients will be $+1$ still while the other $\frac{n}{2}$ will be -1. The particular order of the ± 1's is then $j-$dependent.

In particular, after performing transformation (4.18) and looking at the first system only we see that the reduced density operator os given by $\rho_{\frac{1}{\sqrt{n}}(a^1+a^2+\dots+a^n)} = \mathrm{Tr}_{\bar{1}}(U_H \rho^{\otimes n} U_H^\dagger)$, where by $\mathrm{Tr}_{\bar{1}}$ we denoted the trace over all copies except the 1st one and by U_H the tensor product of local Hadamard transformations. This exactly correspond to the linear combination chosen to make the convergence argument in the 4.2.1. This is why we can argue, that when sending $n \to \infty$ the limiting state will be Gaussian.

By tracing over the different copies we reach the sequences of distribution operators ρ_n for the other Hadamard modes (respective linear combinations of the sets a^i).

Regardless the copy of the system we are looking at we can by linearity of the anti-commutator argue that all of the linear combinations described above fulfill the anti-commutation relations (2.7).

Note that starting from Theorem 4.2 all the statements that have been made are easily translatable into the other possible Hadamard combinations of a^i's. Having in mind that we only consider even states, the intuition given in the Section 4.2.2 and the fact that the signs appearing in other Hadamard combinations do not affect limit in Corollary 4.2 this generalization is straightforward.

In other words we indeed have a limiting Gaussian state as stated in equation (4.3).

4.3 Applications

One application of Theorem 4.1 is in entanglement theory. Entanglement, especially in the many-body setting, is still not very well understood. Attempts of its

quantification lead to the concept of entanglement measure [69]. These quantities are, however, usually very difficult to compute and it is therefore useful to have at least an estimate on the amount of entanglement in the system. Indeed some of the entanglement measures meet the assumption of Theorem 4.1. As shown in [87] the examples would be *distillable entanglement* (see [7]) or *squashed entanglement* [80]. These are measures proposed to quantify an entanglement of bipartite mixed states.

On the other hand if we consider strongly sub-additive functionals instead of strongly super-additive ones we can repeat all the arguments of the Section 4.2 with the only exception which is the opposite inequality in (4.3). This option of upperbounding will be explored in Chapter 5.

In this Chapter we gave the proof of Extremality Theorem for fermions. This theorem tells us that the extremal values of certain class of functionals over the set of all states with the same 2nd moments are reached on the Gaussian states.

The applications in entanglement theory have been pointed out. Further use of this theorem will be made in Chapter 5.

We only gave the proof in Fock space. The representation-independent version of the proof can be easily formulated using our recipe together with the von Neumann uniqueness theorem [13], which says that in finite dimension all the representations of canonical anti-commutation relations are unitarily equivalent to the Fock representation. The map that allows to translate the states and operators from an arbitrary representation to the Fock representation can be found in [51] as well as the representation-independent versions of Theorem 4.2 and Corollary 4.1. For a detailed discussion of the representations of canonical anti-commutation relations we refer to [12].

This is the concluding chapter on the matter of Gaussian states. In the following chapters we will be interested in applying of what we learned about states to the fermionic Gaussian maps.

Chapter 5

Fermionic Quantum Channels

5.1 Quantum channels

Let us consider the time evolution of the system. For closed systems, given the hermitian property of the Hamiltonian, by Stone's theorem [70], time evolution is unitary, i.e. reversible. If however an open system coupling to an inaccesible environment is taken into account it leads to the irreversible time evolution. This is the way how to describe dissipation and decoherence. We would like to introduce a general framework, that will provide a description for every deterministic *physical* time evolution.

Let us call the time evolution in the Schroedinger picture T and define it as a map $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (where by $\mathcal{B}(\mathcal{H})$ we denote the space of bounded linear operators on the Hilbert space $\mathcal{H}$) that has to fulfill the following requirements [86]: linearity, preservation of the trace and complete positivity.

Let us add several further comments on physical meaning of these requirements. First of them, linearity is one of the basic building block of quantum mechanics. [1] In terms of T, the linearity property correspond to the usual

$$T(\alpha A + \beta B) = \alpha T(A) + \beta T(B), \quad \forall A, B \in \mathcal{B}(\mathcal{H}), \forall \alpha, \beta \in \mathbb{C}. \tag{5.1}$$

The next requirement is that we want the map T to be trace preserving, i.e.

$$\mathrm{Tr}[T(A)] = \mathrm{Tr}[A], \forall A \in \mathcal{B}(\mathcal{H}). \tag{5.2}$$

Physically this corresponds to the fact that the probability is preserved, i.e. that the maps is deterministic. There is however one more thing connected to the requirement on $T(A)$ to be density operator. Trace preservation ensures right normalization, but in addition to that we want $T(A)$ to be positive: $T(A^\dagger A) \geq 0, \forall A \in \mathcal{B}(\mathcal{H})$. What we required above was not however only positivity, but so-called complete positivity. We say that map is completely positive if and only if the map

$$T \otimes \mathbb{1}_n \tag{5.3}$$

[1] linearity for instance implies no-cloning [90]

is positive $\forall n$.

We can justify this requirement as follows. Let us imagine that we have system S, the time evolution of which (due to the interaction with some environment) is given by T. If we envision S to be initially entangled with an additional system S' and if T is just a positive (but not completely positive) map, then we can easily end up in the situation that the joint density matrix of S and S' attains negative eigenvalues - a physically meaningless result.

Now we see that the notion of complete positivity is exactly the way how to avoid this situation. In the map (5.3) we require the evolution to be positive for any size of the environment.

For completeness let us now the translate time evolution requirements into the Heisenberg picture. Let us denote the dual map of T by T^*. We can relate T^* via

$$\mathrm{Tr}[T(\rho)A] = \mathrm{Tr}[\rho T^*(A)], \forall \rho, A \in \mathcal{B}(\mathcal{H}). \tag{5.4}$$

Linearity and complete positivity then remains in the form (5.1), (5.3). Trace preservation is now expressed via [84]

$$T^*(\mathbb{1}) = \mathbb{1}. \tag{5.5}$$

This property is also-called unitality.

Definition 5.1. *A map that fulfills the conditions of linearity, complete positivity and trace preservation in the Schroedinger or unitality in the Heisenberg picture is called quantum channel.*

Le us add few more comments on our definition. First of all, being interested in deterministic time evolution only is connected to the trace preservation. The probabilistic time evolution (after the transformation we consider having given state with certain probability only) is indeed physical but not necessarily trace preserving.

Considering the complete positivity requirement, the completely positive time evolution is the only physical evolution only if we assume no correlation (not even the classical ones) in between system and environment prior to applying the channel. If we allow for initial correlation of system and environment (and it does not matter whether classical or quantum) and still attempt to describe the system state at a later time as a function of ots initial state only then there are examples of the physical time evolution that is not completely positive (see [17]).

Before moving to the possible ways of how to represent quantum channels let us recall an important lemma that will allow us to check the complete positivity of the map. We see that the definition (5.3) is little inconvenient for practical purposes. What the following theorem tells us is that instead testing positivity for each size of environment we can just check positivity on cleverly chosen state [86]:

Theorem 5.1. *A linear map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is completely positive if and only if*

$$(T \otimes \mathbb{1}_{\mathcal{H}}) |\Psi\rangle \langle\Psi| \geq 0, \tag{5.6}$$

where $|\Psi\rangle$ *is the maximally entangled state* [2] *on* $\mathcal{H} \otimes \mathcal{H}$.

5.2 Representations of quantum channels

In previous section we have introduced the concept of quantum channel as the most general approach to describe the time evolution of quantum system. Now we are interested in the possible ways of representing a quantum channel. We will introduce the two approaches most relevant for this work. One possibility is to see quantum channel as a dynamics of the system and environment reduced on the system only, as had been suggested in the discussion above. Another option that leads to the duality in between state and channel is to describe the channel in terms of bipartite quantum state.

Let us start with the latter. We can see this duality as a result of the Theorem 5.1, that tells us that checking complete positivity of the map T is equivalent to checking only positivity of the operator $(T \otimes \mathbb{1}) |\Psi\rangle \langle\Psi|$. Let us define so-called Jamiolkowski state and Choi matrix of a quantum channel.

Definition 5.2. *The Jamiolkowski state of the quantum channel* T *is*

$$\tau = (T \otimes \mathbb{1}_{\mathcal{H}}) |\Psi\rangle \langle\Psi| \tag{5.7}$$

and the corresponding Choi matrix

$$d \cdot \tau = d \cdot (T \otimes \mathbb{1}_{\mathcal{H}}) |\Psi\rangle \langle\Psi| , \tag{5.8}$$

where $d = \dim\mathcal{H}$ *and* $|\Psi\rangle$ *is maximally entangled state.*

We emphasize that there is a one-to-one correspondence in between T and τ for all maximally entangled states Ω, $\forall\rho \in \mathcal{B}(\mathcal{H})$, $\forall A \in \mathcal{B}(\mathcal{H})$. In particular, while the map $T \mapsto \tau$ is expressed via (5.7) its inverse $\tau \mapsto T$ is given by [86]

$$\mathrm{Tr}[T(\rho)A] = d \cdot \mathrm{Tr}[\tau A \otimes \rho^T]. \tag{5.9}$$

The correspondence between bipartite states and quantum channels expressed by the equations (5.7) and (5.9) is called *Choi-Jamiolkowski* state-channel duality.

In fact, the Jamiolkowski state provides another representation of the quantum channel in terms of a convex combination of rank-one operators [63]

Theorem 5.2. *A linear map* T *is a quantum channel if and only if it admits the following representation*

$$T(\rho) = \sum_{i=1}^{M} K_i \rho K_i^\dagger \tag{5.10}$$

where the operators K_i *meet the condition*

$$\sum_i K_i^\dagger K_i = \mathbb{1}. \tag{5.11}$$

[2] $|\Psi\rangle$ is of the form $\frac{1}{\sqrt{d}} \sum_i^d |i\rangle |i\rangle$, where $d = \dim\mathcal{H}$; from now on we denote maximally entangled state by $|\Psi\rangle$

The representation of quantum channel introduced in the Theorem 5.2 is called Kraus representation. The conditions (5.10), (5.11) correspond to complete positivity and trace preservation respectively.

As pointed out above there is another, maybe physically more intuitive, point of view on quantum channels, that is to consider a quantum channel as a part of the dynamics of a bigger system :

Theorem 5.3. *Let T be a quantum channel ($T : \mathcal{M}_d \mapsto \mathcal{M}'_d$). Then there is a unitary operator $U \in \mathcal{M}_{dd'^2}$ and a state $\varphi \in \mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$ such that*

$$T(\rho) = \mathrm{Tr}_E[U(\rho \otimes |\varphi\rangle \langle\varphi|)U^\dagger], \tag{5.12}$$

where Tr_E means the trace over the 'environment', i.e., the first two tensor factors in the Hilbert space $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$.

This 'open system' approach was proposed by Stinespring in [77] and it is therefore called the Stinespring representation of the quantum channel. For the completeness let us give a Heisenberg picture version of the Theorem 5.3

Theorem 5.4. *Let T be a quantum channel ($T : \mathcal{M}_d \mapsto \mathcal{M}'_d$). Then for every $r \geq \mathrm{rank}(\tau)$ (where τ is given by (5.7)) there is $V : \mathbb{C}^d \mapsto \mathbb{C}^{d'} \otimes \mathbb{C}^r$ such that V is an isometry, i.e.*

$$V^\dagger V = \mathbb{1}_d \tag{5.13}$$

and for every $A \in \mathcal{M}_{d'}$ is holds

$$T^*(A) = V^\dagger (A \otimes \mathbb{1}_r) V. \tag{5.14}$$

The ancillary space, $\mathbb{C}^r$ in our case, is also called dilation space and the representation (5.4) is then referred to as Stinespring dilation. In case that the dimension of dilation space equals $r = \mathrm{rank}(\tau)$ the dilation space is called minimal. Any two minimal dilations are isometric (see for instance [48]).

Let us conclude this section by defining the so-called complementary channel.[3]

Definition 5.3. *Let T be a quantum channel given by (5.12), then its complementary channel is given by*

$$T^C(\rho) = \mathrm{Tr}_S[U(\rho \otimes |\Psi\rangle \langle\Psi|)U^\dagger], \tag{5.15}$$

where Tr_S denotes the trace over the last of the three tensor factors.

[3]sometimes also called the conjugate channel (cf. [57]), the term complementary is used for example in [48]

5.3 Degradability

The notion of degradability of the channel was introduced by Devetak and Shor [30].

Definition 5.4. *The channel T is called* degradable *if there is another completely positive map W, called the degrading map, such that*

$$W \circ T = T^C. \tag{5.16}$$

The channel T is called anti-degradable *if its complement is degradable, i.e., there is a completely positive map W' such that*

$$W' \circ T^C = T. \tag{5.17}$$

W is called the degrading map.

We can think of degradable channels as of those for which the complementary channel is as least as noisy as the original one. What the degrading map does is adding the noise such that complementary channel is generated.

We see directly from Definition 5.4 how to check whether a given channel is degradable. First it has to hold [25]

$$\mathrm{Ker}T \subseteq \mathrm{Ker}T^C. \tag{5.18}$$

If this is fulfilled we invert the channel T on the complement of its kernel and in accordance with the defining equation (5.16) we check complete positivity of the map

$$W = T^C \circ T^{-1}. \tag{5.19}$$

In order to get more concrete idea of degradability let us quote an easy example of degradability condition for a simple channel.

5.3.1 Example: Degradability of an erasure channel

The erasure channel (see e.g. [49]) acts on the system in the way that with probability $1-p$ a state is mapped onto itself and with probability p it is lost (environment gets it - this is described by the projection on some state $|e\rangle$ that is orthogonal to the all states in the input space of the channel).

Thus we can write the action of the channel on the state ρ

$$T(\rho) = (1-p)\rho + p\,|e\rangle\langle e|\,. \tag{5.20}$$

It is easy to see that the complementary channel then acts as

$$T^C(\rho) = p\rho + (1-p)\,|e\rangle\langle e|\,. \tag{5.21}$$

From the definition of degradability we see that we can construct the degrading map as follows. Let us find an inverse of the channel T: T^{-1}. The degrading

channel must then be equal to $W_{erasure} = T^C \circ T^{-1}$. So we find an explicit form of such a map and can check the complete positivity and normalization. Since the erasure channel is only parameterized by the probability of losing the state to the environment, p we immediately see that

$$T^{-1}(\rho) = \frac{1}{1-p}T(\rho) - \frac{p}{1-p}\left|e\right\rangle\left\langle e\right|. \tag{5.22}$$

Composing this with the complementary channel (5.21) we obtain

$$W_{erasure} = T^C \circ T^{-1} = \frac{p}{1-p}T(\rho) + \frac{1-2p}{1-p}\left|e\right\rangle\left\langle e\right|, \tag{5.23}$$

which is a channel if and only if $p \leq \frac{1}{2}$.

By following the same line of argumentation for the channel T^C instead for T we can conclude that the erasure channel is anti-degradable for $p \geq \frac{1}{2}$.

□

There is one more concept related to degradability and that is so-called weak degradability [20]. In fact, it is a weakened version of degradability and it is a necessary condition for degradability as it has been described above.

For the explanation of this concept and its relation to degradability we will make use of the Stinespring representation of the quantum channel. Let us emphasize again that when introducing the Stinespring representation (5.12) we assumed that the environment was in a pure state. Then the properties of the channel are, up to isometry (in the environment), uniquely determined. This means that having a channel T of the form

$$T(\rho) = \mathrm{Tr}_E[U(\rho \otimes \left|\Psi\right\rangle\left\langle\Psi\right|)U^\dagger], \tag{5.24}$$

its properties are independent of the choice of possible U's and Ψ's. We will discuss some of the properties of physical interest in the following section. Let us now consider the situation that we would be for some purposes willing to give up the environment independence.

In that case, we can consider to represent the channel in the form of (5.12), but with the mixed environmental state (for the price of uniqueness, of course). We can then write

$$T(\rho) = \mathrm{Tr}_E[U(\rho \otimes \sigma)U^\dagger], \tag{5.25}$$

where ρ refers to the state of the system, σ is the (generally mixed) state of the environment. Then there is a natural way of defining the complementary channel as

$$T^C(\rho) = \mathrm{Tr}_E[U(\rho \otimes \sigma)U^\dagger]. \tag{5.26}$$

The definition of weak degradability is then the following [20].

Definition 5.5. *A channel T is weakly degradable with respect to the representation* (5.25) *if and only if there is a channel W such that*

$$(W \circ T)(\rho) = T^C \rho \tag{5.27}$$

for all input states ρ.

5.4 Covariance of quantum channel

Let us now make a short note on another special property of quantum channels that is called covariance. We will see in the upcoming sections that (as degradability) covariance may be used to simplify significantly the evaluation of some channel properties that would be otherwise extremely hard to compute.

The concept of covariance was firstly introduced in this context in [47] and [23]. At this point let us only state the precise definition of this quantum channel property. Let

$$\mathcal{S} = \{U_1, \ldots, U_d\} \tag{5.28}$$

be a set of the unitary operators on $\mathcal{H}$ that completely randomize any quantum state, i.e., for any state $\rho \in \mathcal{B}(\mathcal{H})$ and U_j's from $\mathcal{S}$ we have

$$\frac{1}{d}\sum_{j=1}^{d} = U_j \rho U_j^\dagger = \frac{\mathbb{1}}{\dim\mathcal{H}}. \tag{5.29}$$

Having $\mathcal{S}$ we can define covariant channels as follows

Definition 5.6. *A quantum channel T is $\mathcal{S}$-covariant if and only if it commutes with all operators U from the set $\mathcal{S}$, i.e.,*

$$T(UAU^\dagger) = UT(A)U^\dagger, \forall U \in \mathcal{S} \tag{5.30}$$

This property will be useful for computing the channel capacities in the next section.

5.5 Capacities of quantum channels

One of the most important quantities associated with (both quantum and classical) channels is their capacity to transmit information. While in the classical case a single number is enough to characterize a channel, in the quantum case there is a variety different quantities depending on the required task and the possible availability of additional resources.

The channel capacities have been a subject of broad interest in quantum information theory. Here we only introduce a few basic notions of capacity theory and refer for broader review to [85]. We discuss the strong impact that the assumption of degradability has on the evaluation of the channel capacity.

5.5.1 Classical capacity of quantum channel

The first important quantity of interest is the so-called classical capacity of the quantum channel, i.e. how much classical information can be transmitted faithfully through the channel per use. In particular, let us suppose that the sender (Alice) wants to send a message consisting of n classical bits to a receiver (Bob) via quantum channel. In case that Alice is only allowed to use the channel m times we can define an achievable rate of the channel as follows [14].

Definition 5.7. *A number R is called an achievable rate for a quantum channel T if for any $\epsilon > 0$ and any $\delta > 0$ there is a transmission protocol sending n bits over m uses of the channel such that $\frac{n}{m} > R - \delta$ with error probability that is smaller than ϵ.*

The classical capacity of the quantum channel is then defined as

Definition 5.8. *The classical capacity of quantum channel is the smallest number C such that $R \leq C$ for any achievable rate (i.e., supremum of all achievable rates).*

The calculation of C for a given channel is very difficult talk. For that purpose, let us start with definition of so-called one-shot capacity:

Definition 5.9. *The classical one-shot capacity of quantum channel is the minimum number C_1 such that $R \leq C_1$ for any achievable rate with additional restriction being that Alice can only use product states for signaling.*

The relation between C and C_1 is given by

$$C(T) = \lim_{m\to\infty} \frac{C_1(T^{\otimes m})}{m}. \tag{5.31}$$

It has been proved in[43] that in general it *does not* hold that

$$C(T) = C_1(T). \tag{5.32}$$

The one-shot capacity can be computed using Holevo-Schumacher-Westmoreland theorem (see [73],[46])

$$C_1(T) = \sup_{\rho_E} \chi(\rho_E), \tag{5.33}$$

where ρ_E is the average ensemble state

$$\rho_E = \sum_i p_i \rho_i, \rho_i \in \mathrm{Im}(T) \tag{5.34}$$

and $\chi(\rho_E)$ so-called Holevo quantity:

$$\chi(\rho_E) = S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i) \tag{5.35}$$

with S being the von Neumann entropy

$$S(\xi) = \mathrm{Tr}(\xi \log \xi). \tag{5.36}$$

There are only very few quantum channels for which the classical capacity is known, for example one-qubit unital product channels [56], bosonic attenuation channels [39] and fermionic attenuation channels [14], where the last example is of especial relevance to this work. Let us list the main features of this channel as an example.

5.5.2 Example: Classical capacity of fermionic product channels

Fermionic product channels have been introduced in [14] to describe lossy transmission of fermions (lossy attenuation channels). In particular, under fermionic product channel we understand a class of channels that are expressible as a product of fermionic attenuation channels. In particular, using notation (2.15)

$$c(\mathbf{x}) = c_1^{x_1} c_2^{x_2} \dots c_{2n}^{x_{2n}}, \tag{5.37}$$

where $c_i, i \in \{1, 2, \dots 2n\}$ are Majorana fermions defined via (2.8) and $\mathbf{x} = (x_1, x_2, \dots, x_{2n})$ is a binary string of $2n$ bits.

Fermionic product channel T acts as

$$T(\mathbb{1}) = \mathbb{1} \tag{5.38}$$

$$T(c_i) = b_i c_i, \quad i = 1, \dots, 2n \tag{5.39}$$

$$T(c(\mathbf{x})) = \prod_{i:x_i=1} b_i c(\mathbf{x}), \tag{5.40}$$

where $b_1, \dots b_{2n}$ are real parameters specifying the particular channel with an additional property $0 \leq b_i \leq 1, \forall i$ (therefore the name product of attenuation channels).

First relevant property of this channel is that it maps Gaussian states on Gaussian states. To see this let us first define another fermionic product channel acting on $n+1$ qubits

$$\tilde{T}(c_i) = b_i c_i, i = 1, \dots, 2n, \tag{5.41}$$

$$\tilde{T}(c_{2n+1}) = c_{2n+1}, \tag{5.42}$$

$$\tilde{T}(c_{2n+2}) = 0. \tag{5.43}$$

Let us also recall a correspondence of the set of all fermionic Gaussian with its even parity subset (see Section 2.4.2) realized by map

$$\mathcal{E}(\rho) = V\rho \otimes \frac{\mathbb{1}}{2} V^\dagger, \tag{5.44}$$

where ρ is general Gaussian state and V is unitary operator. In [14] Bravyi argues that product fermionic channels (5.38) - (5.40) and (5.41) - (5.43) commute with the parity map $\mathcal{E}$ in the following sense

$$\mathcal{E} \circ T = \tilde{T} \circ \mathcal{E}. \tag{5.45}$$

Using this equality we argue about a Gaussianity of the output of the channel T acting on Gaussian input: let us consider Gaussian state ρ, then we know from the section 2.4.2 that $\mathcal{E}(\rho)$ is even Gaussian state and, by definition of $\tilde{T}$, $\tilde{T} \circ \mathcal{E}(\rho)$ is as well. Then by (5.45) we conclude that $\mathcal{E} \circ T(\rho)$ is even Gaussian as well, which implies that $T(\rho)$ is Gaussian state (again, for the details see Section 2.4.2). In case that we would impose the even parity requirement at the beginning, the proof is of course trivial.

Let us now add a few more comments on the properties of the channel (5.38) - (5.40). First, note that the moments of ρ and $T(\rho)$ are related via

$$\mathrm{Tr}(T(\rho)c((x))) = \prod_{i:x_i=1} b_i \mathrm{Tr}(\rho c(\mathbf{x})). \tag{5.46}$$

By Wick's theorem (Theorem 2.1)

$$\mathrm{Tr}(\rho c(\mathbf{x})) = i^l \mathrm{Pf}(M[\mathbf{x}]), \tag{5.47}$$

so we see that on the level of covariance matrices the channel output is related to the input by a simple diagonal transformation

$$T(\Gamma) = B\Gamma B^T, \quad \text{with} \quad B = \mathrm{diag}(b_1, \ldots, b_{2n}). \tag{5.48}$$

Next point is that the channel (5.38) - (5.40) is covariant (see Section 5.4). Let the set $\mathcal{S}$ (see (5.28)) be constituted from 4^n operators $c(\mathbf{x})$. Using the definition (5.37) we get

$$4^{-n} \sum_{\mathbf{x}} c(\mathbf{x})c(\mathbf{y})c(\mathbf{x}) = 0, \forall y \neq 0, \tag{5.49}$$

$$4^{-n} \sum_{\mathbf{x}} c(\mathbf{x})c(\mathbf{y})c(\mathbf{x}) = \mathbb{1}, y = 0. \tag{5.50}$$

which means that we found the set with the property (5.29). Now we only have to check the defining equation of the covariance property (5.30). But again it is straightforward from the definition (5.38) - (5.40) that for an arbitrary operator X it holds

$$T(c(\mathbf{x})Xc(\mathbf{x})) = c(\mathbf{x})T(X)c(\mathbf{x}). \tag{5.51}$$

It is proven in [47] that the following one-shot capacity formula holds for covariant n-qubit channels

$$C_1 = n - S_{\min}(T), \tag{5.52}$$

where $S_{\min}(T)$ is the minimum output entropy of the channel Tgiven by

$$S_{\min}(T) = \min_{\rho} S(T(\rho)). \tag{5.53}$$

Let us give a short proof of this statement [14]. Let $|\Phi\rangle$ denote the state with the minimal output entropy. If we choose U's from the set $\mathcal{S}$ then indeed an ensemble $\{U|\Phi\rangle\}$ has maximally mixed average state

$$\rho_E = \frac{1}{d} \sum_{i=1}^{d} U_i |\Phi\rangle \langle\Phi| U_i^\dagger = \frac{1}{2^n} \mathbb{1}. \tag{5.54}$$

Then, for the Holevo quantity (5.35) we immediately obtain

$$\chi(\rho_E) = n - S(T(|\Phi\rangle\langle\Phi|)) = n - S_{\min}(T). \tag{5.55}$$

We see that this is the maximal value of Holevo quantity χ for an arbitrary ensemble since we have simultaneously maximized the first and minimized the second term in equation (5.35). We can therefore conclude that the single-shot capacity of the covariant channel is indeed given by the formula (5.52).

In addition to that, it is proven in [14] that the minimum output entropy of the channel (5.38) - (5.40) is given by

$$S_{\min}(T) = \sum_j H\left(\frac{1 + b_{2j-1}b_{2j}}{2}\right), \tag{5.56}$$

where H is the Shannon entropy. This solves the classical capacity problem of fermionic product channels.

□

5.5.3 Quantum capacity

Let us now consider that we want to transmit quantum information through a quantum channel. Then we talk about the quantum capacity of a quantum channel [76], which corresponds to the maximum number of qubits that can be transmitted faithfully through the channel per use. More precisely:

Definition 5.10. *The quantum capacity of the quantum channel T is given by*

$$Q = \sup \lim_{n\to\infty} \frac{\log_2 D(n)}{n}, \tag{5.57}$$

where the supremum is taken over all subspaces $S(n)$ of the input Hilbert space $(\mathcal{H}^d)^{\otimes n}$ satisfying

$$\lim_{n\to\infty} \int_{|\Psi\rangle\in S(n)} \langle\Psi| T^{\otimes n}(\langle\Psi|\,|\Psi\rangle) d\,|\Psi\rangle = 1 \tag{5.58}$$

and $D(n) = \dim S(n)$.

The equation (5.58) is called average fidelity condition. Also we define the quantum capacity without allowing any classical communication in between a sender and a receiver. In quantum information theory the cases with forward, backward or two-way classical communication are also explored (see cf. [85]). In this work, only the capacity given by Definition 5.10 will be of interest.

It is shown in [76] that the quantum capacity Q of a quantum channel T is given by

$$Q(T) = \lim_{n\to\infty} \frac{1}{n} \sup_{\rho} J(\rho, T^{\otimes n}) \tag{5.59}$$

with

$$J(\rho, T) = S(T(\rho)) - S((T \otimes \mathrm{id})(\Psi)), \tag{5.60}$$

where Ψ is a purification of ρ. The quantity J is called the *coherent information*.

For a general quantum channel the evaluation of the coherent information (and thus the quantum capacity) is rather complicated task because coherent information J is neither concave nor additive (lack of additivity means that it is necessary to consider the regularization of J, lack of concavity, that if a maximum is found numerically it does not have to necessarily be the global one). Nevertheless, there is a class of channels for which the evaluation of quantum capacity can be significantly simplified. These are the degradable channels introduced in Section 5.3.

Using Definition 5.4, let us denote

$$\tilde{\rho}_S = W \circ T(\rho) \tag{5.61}$$

the image of the degrading map W composed with the channel T acting on the input state ρ, i.e, the complementary channel. Let us denote the environment of the degrading channel by E_Ψ and construct the extension of $\tilde{\rho}_S$ to E_W. This extension will be denoted by $\tilde{\rho}_{SW}$.

It is shown in [30] that in case of degradable channels the coherent information (5.60) can be expressed as

$$J(\rho, T) = S(\tilde{\rho}_{SW}) - S(\tilde{\rho}_S), \tag{5.62}$$

where by S we again denote the von Neumann entropy. The RHS of the equation (5.62) is precisely the definition of the conditional entropy [68]:

$$S(E_W|S) = S(\tilde{\rho}_{SW}) - S(\tilde{\rho}_S). \tag{5.63}$$

Let us emphasize that the conditional entropy is strongly sub-additive [68]:

Definition 5.11. *Let ρ be a map acting on $(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) \otimes (\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2})$ and ρ_i restrictions of ρ on $\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$. We say that the function f is strongly sub-additive if and only if it holds that $f(\rho) \leq f(\rho_1) + f(\rho_2)$ (with natural generalization to larger number of parties).*

In case of (5.63) we therefore get

$$S(E_W^{12}|S^{12}) \leq S(E_W^1|S^1) + S(E_W^2|S^2), \tag{5.64}$$

where we refer to the respective subsystems by superscript.

Let us now consider the capacity of degradable channels in the context of the Extremality theorem (see Chapter 4). We are now in the position where we expressed the coherent information, an essential quantity for calculation of quantun

capacity, in terms of the conditional entropy, that is indeed invariant under local unitaries, (trace norm) continuous and strongly sub-additive.

We can therefore apply the Extremality theorem to express capacities of degradable Gaussian fermionic channels. Analogously to the bosonic case worked out in [88] we can write for every Gaussian channel T

$$J(\rho, T) \leq J(\rho_G, T), \tag{5.65}$$

where ρ_G is the Gaussian state with the same second moments as ρ. This leads to the following simplification of Shor's definition of quantum capacity (5.59). For degradable Gaussian channel we can write:

$$Q(T) = \sup_{\rho_G} J(\rho_G, T). \tag{5.66}$$

For a degradable product channel $T = \otimes T_i$ we can obtain

$$J(\rho, \otimes_i T_i) \leq \sum_i J(\rho_i, T_i) \leq \sum_i J(\mathcal{G}(\rho_i), T_i), \tag{5.67}$$

where $\mathcal{G}(\rho_i)$ is Gaussian state with the same 2nd moments as ρ_i. The first inequality in the equation (5.67) follows from sub-additivity of conditional entropy (5.64) and degradability and the second one follows by Extremality theorem for Gaussian degradable channels.

Consequently for the degradable Gaussian product channels we can write

$$Q(\otimes_i T_i) = \sum_i \sup_{\rho_G} J(\rho_G, T_i). \tag{5.68}$$

5.6 Fermionic Gaussian Channels

In this Section we would like to apply the quantum channel framework introduced in this chapter to the fermionic Gaussian states. Our goal is now to construct the channel that would map fermionic Gaussian states on fermionic Gaussian states. We have already seen one example of this class: the fermionic product channel described in the Section 5.5.2.

We will give a general form of fermionic Gaussian channels and discuss the properties outlined in the previous sections: degradability, antidegradability and capacity.

In order to construct channels that map general Gaussian states on Gaussian states we need to extend (2.68) on system+environment and allow interactions. In the following section we will simplify the class of these extended transformations as much as possible without disturbing the essential interaction part by removing local unitary pre- and post-processing.

5.6.1 Simplification of the general form of the quasi-free channel

In Chapter 2 we discussed that the allowed transformations mapping covariance matrices and covariance matrices are special orthogonal transformations (for N modes system $SO(2N)$ transformation (2.62)).

If we consider system-environment point of view at quatum channel we can write the general orthogonal transformation acting at both system and environment in the block form:

$$\begin{pmatrix} R_S & R_{SE} \\ R_{ES} & R_E \end{pmatrix}, \tag{5.69}$$

where R_S refers to the transformation performed on the system only, R_E the one performed on the enviroment only and R_{SE} and R_{ES} refers to interactions system-environment, environment system respectively.

It is desirable to simplify this general transformation as much as possible. As discussed above this the most general form of the transformation. In particular, the local pre- and post-processing is included. We would like to find the simplest form that describes system-environment interaction. For this purpose the local operations can be safely omitted.

Let us first recall the singular value decomposition [68]:

Theorem 5.5. *Let A be a square matrix. Then there exist unitary matrices G, H and a diagonal matrix D with non-negative entries such that $A = GDH$.*

Since we are working with orthogonal (real) transformations let us point out the following Corollary of Theorem (5.5).

If the matrix A to be decomposed is real then the unitaries G, H are also real, meaning they are orthogonal:

$$A = GDH, \qquad \text{A is real} \Rightarrow \quad G, H \text{ are orthogonal.} \tag{5.70}$$

Note that while in general the singular value decomposition can be chosen such that the matrix D is non-negative, in our case we do not have all the orthogonal transformation available. As it has been discussed in Chapter 2 only the special orthogonal transformations are valid transformations on the level of covariance matrices and correspond to the unitaries on the Fock space. This leads to the conclusion that in the most general case the matrix D achieved by means of the special orthogonal transformation can have (at most) one block with negative entries (which occurs if $\det A < 0$).

We will make use of Theorem 5.5 in order to exclude the local operations in the following way:

$$\begin{pmatrix} U_S & 0 \\ 0 & U_E \end{pmatrix} \cdot \begin{pmatrix} R_S & R_{SE} \\ R_{ES} & R_E \end{pmatrix} \cdot \begin{pmatrix} V_S & 0 \\ 0 & V_E \end{pmatrix}. \tag{5.71}$$

Let us first assume that $\dim R_S = \dim R_E$

Now, by choosing transformations U_S, V_S such that they are inverse of those in the singular value decomposition of system transformation R_S and U_E, V_E are the inverse of the orthogonal transformations in the singular value decomposition of the environment transformation R_E, i.e.:

$$\begin{aligned} R_S = G_S D_S H_S \quad &\Rightarrow \quad G_S = U_S^{-1}, H_S = V_S^{-1} \\ R_E = G_E D_E H_E \quad &\Rightarrow \quad G_E = U_E^{-1}, H_E = V_E^{-1} \end{aligned} \tag{5.72}$$

Performing multiplication in (5.71) using (5.72) we obtain

$$\begin{pmatrix} D_S & U_S R_{SE} V_E \\ U_E R_{ES} V_S & D_E \end{pmatrix}. \tag{5.73}$$

The orthogonality required of matrix (5.73) gives two conditions to be fulfilled

$$\begin{pmatrix} D_S & U_S R_{SE} V_E \\ U_E R_{ES} V_S & D_E \end{pmatrix} \cdot \begin{pmatrix} D_S^T & V_S^T R_{ES}^T U_E^T \\ V_E^T R_{SE}^T U_S^T & D_E^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.74}$$

and

$$\begin{pmatrix} D_S^T & V_S^T R_{ES}^T U_E^T \\ V_E^T R_{SE}^T U_S^T & D_E^T \end{pmatrix} \cdot \begin{pmatrix} D_S & U_S R_{SE} V_E \\ U_E R_{ES} V_S & D_E \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.75}$$

Equations (5.82) and (5.83) can be used to infer the effect of the local singular decomposition transformation on the system-eviroment R_{SE} and environment-system parts of the transformation (5.69). In particular, if denote $\tilde{R}_{ES} = U_S R_{SE} V_E$ and $\tilde{R}_{ES} = U_E R_{ES} V_S$ it follows from (5.82)

$$\tilde{R}_{SE} \tilde{R}_{SE}^T = 1 - D_S^2. \tag{5.76}$$

On the other hand $\tilde{R}_{SE}$ also has singular value decomposition

$$\tilde{R}_{SE} = W_1 D_{SE} W_2, \tag{5.77}$$

where W_1, W_2 are orthogonal matrices and D_{SE} is diagonal matrix with non-negative entries. So we can express the LHS of (5.76) as

$$\tilde{R}_{SE} \tilde{R}_{SE}^T = W_1 D_{SE}^2 W_1^T. \tag{5.78}$$

By comparison with (5.76) we can conclude that W_1 is an identity matrix: $W_1 = \mathbb{1}$.

Obviously using the equation (5.83) we can the in the exactly same manner determine the form of W_2. In this case we obtain the condition on the expression $\tilde{R}_{SE}^T \tilde{R}_{SE}$:

$$\tilde{R}_{SE}^T \tilde{R}_{SE} = 1 - D_E^2. \tag{5.79}$$

Using the singular value decomposition (5.77) we obtain

$$\tilde{R}_{SE}^T \tilde{R}_{SE} = W_2^T D_{SE}^2 W_2. \tag{5.80}$$

Combining (5.79) and (5.80), similarly like in the previous case, we conclude that W_2 must be identity. By uniqueness of singular value decomposition we can then argue that

$$\tilde{R}_{SE} = W_1\sqrt{1-D_E^2}W_2 = Y\sqrt{1-D_E^2}, \tag{5.81}$$

where $Y = \mathrm{diag}\{\pm 1, \ldots, \pm 1\}$.

We are therefore at the point, where we know three out of four elements of the desired transformation, let us denote the fouth one by X. The orthogonality conditions now therefore have the form

$$\begin{pmatrix} D_S & Y\sqrt{1-D_S^2} \\ X & D_E \end{pmatrix} \cdot \begin{pmatrix} D_S^T & X^T \\ Y\sqrt{1-D_S^2}^T & D_E^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.82}$$

and

$$\begin{pmatrix} D_S^T & x^T \\ Y\sqrt{1-D_S^2} & D_E^T \end{pmatrix} \cdot \begin{pmatrix} D_S & Y\sqrt{1-D_S^2} \\ X & D_E \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.83}$$

By direct inspection of matrix elements we conclude that $X = -Y'\sqrt{1-D_E^2}$.

We can therefore conclude that singular value decompostion of system and environment parts of the transformations R_S and R_E insures that interaction parts of the transformation R_{SE} and R_{ES} are diagonalized as well.

Based on these results the transformation (5.69) now has the form:

$$\begin{pmatrix} D_S & Y\sqrt{1-D_S^2} \\ -Y'\sqrt{1-D_E^2} & D_E \end{pmatrix}, \tag{5.84}$$

which can be directly generalized for the different dimension of R_S nad R_E. Further we notice that we can infer that $D_S = D_E$ in case of equal dimensions of system and environment with straightforward modification for the different dimensions. Note that the possible sign modification of D_E with respect to D_S cancels due to the orthogonality condition on the matrix (5.69). We will provide this generalization in Section 5.8.

Let us emphasize that physically this correspond to sort of symmetry of interaction in between system and environment. If the off-diagonal elements of the matrix (5.69) are zero, then we have identity on both system and environment, but at the moment there is any interaction, as soon as some transformation is done on the environment, it is immediately reflected on the system and vice versa.

Let us first treat the case of $\mathrm{dim}\Gamma_S = \mathrm{dim}\Gamma_E$, i.e. from now on we can write $D_S = D_E$. We conclude the discussion above for this case in the following Theorem.

Theorem 5.6. *Let Γ_S, Γ_E be covariance matrices describing the fermionic Gaussian state of the system and environment respectively. Let us assume that* $\mathrm{dim}\Gamma_S = \mathrm{dim}\Gamma_E$. *General Gaussian transformation of such a system, described by the matrix* (5.69)*, can be simplified by removing all the local parts into the following standard form*

$$\begin{pmatrix} D_S & Y\sqrt{1-D_S^2} \\ -Y\sqrt{1-D_S^2} & D_S \end{pmatrix}. \tag{5.85}$$

5.6.2 Transforming Gaussian states

Making use of all the freedom available to simplify the channel by excluding local pre- and post-processing and physically irrelevant permutation of the modes we can now investigate the action of such map on general Gaussian state of the system of the interest and environment with the covariance matrix $\Gamma_S \oplus \Gamma_E$:

$$\Gamma = \begin{pmatrix} \Gamma_S & 0 \\ 0 & \Gamma_E \end{pmatrix}. \tag{5.86}$$

Let us write the action of the channel T under consideration on the level of covariance matrices

$$T(\Gamma) = \mathrm{Tr}_E(U\Gamma U^\dagger). \tag{5.87}$$

The meaning of the notation $\mathrm{Tr}_E(X)$ here is that we take the block of the matrix X corresponding to the modes not in E. Using analogous notation we can write the complementary channel as

$$T^C(\Gamma) = \mathrm{Tr}_S(U\Gamma U^\dagger). \tag{5.88}$$

Explicitly we can express the action of these via adjoint action of (5.85) on (5.86). The result for equal dimension of system and environment is summarized in the following theorem.

Theorem 5.7. *Let Γ_S, Γ_E denote the covariance matrices characterizing the state of the system and environment respectively. Let $\mathrm{dim}\Gamma_S = \mathrm{dim}\Gamma_E$. Then the fermionic Gaussian channel* (5.87) *has the standard form*

$$T: \quad \Gamma_S \mapsto D_S\Gamma_S D_S + \sqrt{1-D_S^2}Y\Gamma_E Y\sqrt{1-D_S^2} \tag{5.89}$$

while the standard form of its complement (given in general by (5.90)*) reads*

$$T^C: \quad \Gamma_S \mapsto \sqrt{1-D_S^2}Y\Gamma_S Y\sqrt{1-D_S^2} + D_S\Gamma_E D_S. \tag{5.90}$$

Let us recall that the channel T is called *degradable* if there is another completely positive map W such that

$$W \circ T = T^C. \tag{5.91}$$

Now we would like analyze degradability of the fermionic Gaussian channels on the level of covariance matrices. Before proceeding to the detailed analysis let us add an example regarding the gauge-invariance of the transformation.

Even when removing all local operations from the system+environment transformation the in general non-gauge-invariant nature of the transformation is preserved. We would like to illustrate that, at least, in the low mode cases, it is possible to avoid to the non-gauge-invariant transformations by changing (by the non-Gaussian) means the environmental state.

5.6.3 Example: Gauge-invariance of Hamiltonian vs. environmental state

Let us consider the Hamiltonian

$$H = i(ab^\dagger + a^\dagger b), \tag{5.92}$$

where $a, a^\dagger$ are annihilation and creation operators for the mode of the system and $b, b^\dagger$ are annihilation and creation operator for the environmental mode. The unitary time evolution operator then reads

$$U = \exp(-i\theta H) = \exp[\theta(ab^\dagger + a^\dagger b)]. \tag{5.93}$$

Let us consider the state of the system to be $|1\rangle$, i.e. the mode under consideration is occupied and the state of the environment to be $|0\rangle$, i.e., the respective mode is unoccupied. The overall product state will be denoted $|10\rangle$. Next we will consideet the time evolution of the state governed by the Hamiltonian (5.92):

$$\begin{aligned} U|10\rangle = Ua^\dagger|00\rangle = Ua^\dagger U|00\rangle = -(\cos\theta a^\dagger + i\sin\theta b^\dagger)|00\rangle = \\ = -\cos\theta|10\rangle - i\sin\theta|01\rangle . \end{aligned} \tag{5.94}$$

We made use of Baker-Hausdorff identity [68]:

$$\exp(\lambda G)A\exp(-\lambda G) = \sum_{n=o}^{\infty}\frac{\lambda^n}{n!}C_n, \tag{5.95}$$

where C_n is defined in recursive manner via the commutators: $C_0 = A$, $C_1 = [G, C_0]$, $C_2 = [G, C_1], \ldots C_n = [G, C_{n-1}]$. Here λ denotes a complex number and $A, G, C_n, \forall n$ are operators. Since we want to compute the time evolution of the annihilation operator a: $Ua^\dagger U$, in the case of Hamiltonian (5.92) we therefore set $G := (ab^\dagger + a^\dagger b)$ and commutators will give us $[G, a] = -b$, $[G, [G, a] = -a$ etc. In particular all the odd terms in the sum (5.95) will be proportional to $-b$ and all the even to $-a$. After performing the summation we obtain

$$UaU^\dagger = -\cos\theta a - i\sin\theta b. \tag{5.96}$$

Analogously we can evaluate the time evolution of $a^\dagger, b, b^\dagger$.

Up to this point we obtained time evolution (5.94) by using a gauge-invariant Hamiltonian (5.92), while starting with one particle in system mode and an empty environmental mode. Let us now consider gauge-non-invariant Hamiltonian

$$H_N = i(ab + a^\dagger b^\dagger), \tag{5.97}$$

that gives the time evolution

$$U_N = \exp[\theta(ab + a^\dagger b^\dagger)] \tag{5.98}$$

In addition to that we will add a particle to environmental mode, i.e. we will start off the state $|11\rangle$. Let us now explore the time evolution of this state governed by Hamiltonian (5.97) in exactly the same manner as we did for gauge-invariant one. In the case G will set $ab + a^\dagger b^\dagger$. And commutators will result into $[G, a] = -b^\dagger$, $[G, [G, a]] = -a$ etc. After summation in (5.95) we obtain

$$UaU^\dagger = -\cos\theta a - i\sin\theta b^\dagger. \tag{5.99}$$

The only thing left now is to find an explicit expression for the time evolution. While in the gauge-invariant case we implicitly used $U|00\rangle = |00\rangle$, which indeed holds for the unitary (5.93) while for the unitary (5.98) we get

$$U_N|00\rangle = \cos\theta|00\rangle + i\sin\theta|11\rangle, \tag{5.100}$$

so for $|00\rangle$ we have

$$|00\rangle = \frac{1}{\cos\theta}U_N|00\rangle - i\tan\theta|00\rangle. \tag{5.101}$$

The resulting time evolution is therefore

$$\begin{aligned} U_N|11\rangle &= \cos^{-1}\theta U_N a^\dagger U_N^\dagger b^\dagger U_N|00\rangle - i\tan\theta U_N a^\dagger U_N b^\dagger|11\rangle = \\ &= \cos^{-1}\theta(-\cos\theta a^\dagger - i\sin\theta b)(-\cos\theta b^\dagger - i\sin\theta a)|00\rangle = \\ &= \cos^{-1}\theta(\cos^2\theta a^\dagger b^\dagger + i\sin\theta\cos\theta a^\dagger a + i\sin\theta\cos\theta bb^\dagger + \sin^2\theta ba)|00\rangle = \\ &= \cos\theta|11\rangle + i\sin\theta|00\rangle. \end{aligned} \tag{5.102}$$

It is easy to see that after tracing the environment we get in both (5.94) and (5.102) the following time evolution of the state of the system

$$|1\rangle \mapsto \cos^2\theta|1\rangle\langle 1| + \sin^2\theta|0\rangle\langle 0|. \tag{5.103}$$

In conclusion, by merely changing the environmental state we manage to achieve the same transformation of the input state, once using the time evolution governed by gauge-invariant Hamiltonian, once using gauge non-invariant one. Note however that the change of the environmental required prior to the whole procedure was not a Gaussian map.

□

5.7 Weak degradability of fermionic Gaussian channels

Now we would like to formulate conditions on degradability of the channel (5.89). From the definition of degradability (5.16) we see that, provided the channel under consideration is invertible, we get the degrading map W in the following manner. First,we find the inverse of the channel T^{-1}. Then we construct the map $T^C \circ T^{-1}$ and check when of if at all is this map completely positive and trace preserving.

Let us however point out that for the moment we only treat the environment and system of the same dimension. It follows from the discussion in Section 5.2 and Section 5.3 that with this dimensional restriction we can only find the conditions for the fermionic Gaussian channel to be weakly-degradable. Let us give the condition for weak-degradability first and only then treat the generalization to the systems of an arbitrary dimension. Easy computation leads

$$W(\Gamma_S) = \frac{\sqrt{1-D_S^2}}{D_S} Y \Gamma_S Y \frac{\sqrt{1-D_S^2}}{D_S} - \frac{1-D_S^2}{D_S}\Gamma_E \frac{1-D_S^2}{D_S} + D_S \Gamma_E D_S. \quad (5.104)$$

The expression (5.104) is well defined because we can assume that all elements of D_S must be non-zero. In particular, zero entries in D_S would imply that $T(\Gamma_S)$ does not depend on certain elements of Γ_S, whilst $T^C(\Gamma_S)$ depends on them. Hence there can not be degrading map $W(T(\Gamma_S)) = T^C(\Gamma_S)$. From now on let us therefore assume that $0 < |D_S| \leq \mathbb{1}$.

5.7.1 Full weak-degradability characterization

For linear transformations on the level of covariance matrices we have if and only if condition available for checking whether the map of our interest indeed is a channel. This condition is given by the Theorem 3.3.

Comparing our degrading map (5.104) with the general form of the Gaussian trace preserving map (3.37) we immediately see the following correspondence

$$B = Y \frac{\sqrt{1-D_S^2}}{D_S}, \quad (5.105)$$

$$A = -\frac{1-D_S^2}{D_S}\Gamma_E \frac{1-D_S^2}{D_S} + D_S \Gamma_E D_S. \quad (5.106)$$

The condition to be met in our case therefore read

$$M^T M = \begin{pmatrix} -\frac{1-D_S^2}{D_S}\Gamma_E \frac{1-D_S^2}{D_S} + D_S\Gamma_E D_S & Y\frac{\sqrt{1-D_S^2}}{D_S} \\ -Y\frac{\sqrt{1-D_S^2}}{D_S} & 0 \end{pmatrix} \cdot \\ \cdot \begin{pmatrix} -\frac{1-D_S^2}{D_S}\Gamma_E^T \frac{1-D_S^2}{D_S} + D_S\Gamma_E^T D_S & -Y\frac{\sqrt{1-D_S^2}}{D_S} \\ Y\frac{\sqrt{1-D_S^2}}{D_S} & 0 \end{pmatrix} \leq \mathbb{1}. \quad (5.107)$$

Note that this is equivalent to the condition on the eigenvalues of the matrix $M^T M$ to be in the interval $[-1, 1]$. From this condtion ot is still rather direct to check, given the transformation (and therefore D_S) and the environmental states (and therefore Γ_E) whether the channel is degradable or not. Let us conclude this by the theorem.

Theorem 5.8. *Let Γ_S be covariance matrix of the input state and Γ_E covariance matrix of the environmental state of n-mode fermionic Gaussian channel of the form*

$$\Gamma_S \mapsto D_S \Gamma_S D_S + \sqrt{1 - D_S^2} Y \Gamma_E Y \sqrt{1 - D_S^2}. \tag{5.108}$$

Then the channel (5.108) *is weakly degradable (with respect to Γ_E) if and only if the matrix*

$$\begin{pmatrix} -\frac{1-D_S^2}{D_S}\Gamma_E\frac{1-D_S^2}{D_S} + D_S\Gamma_E D_S & Y\frac{\sqrt{1-D_S^2}}{D_S} \\ -Y\frac{\sqrt{1-D_S^2}}{D_S} & 0 \end{pmatrix} \tag{5.109}$$

meets the condition

$$M^T M \leq \mathbb{1}. \tag{5.110}$$

5.7.2 Example: Weak degradability of fermionic attenuation channels

Let us apply Theorem 5.8 to the fermionic product channels introduced in the Section 5.5.2. On the level of covariance matrices fermionic product channel acts as

$$T(\Gamma) = B\Gamma B^T, \quad \text{with} \quad B = \mathrm{diag}(b_1, \ldots, b_{2n}). \tag{5.111}$$

As in [14] we consider the environment be in the maximally mized state, i.e. $\Gamma_E = 0$. In that case matrix (5.109) has the form

$$M = \begin{pmatrix} 0 & \frac{\sqrt{1-D_S^2}}{D_S} \\ -\frac{\sqrt{1-D_S^2}}{D_S} & 0 \end{pmatrix}. \tag{5.112}$$

The condition (5.110) then reads

$$M^T M = \begin{pmatrix} \frac{1-D_S^2}{D_S} & 0 \\ 0 & \frac{1-D_S^2}{D_S} \end{pmatrix} \leq \mathbb{1}. \tag{5.113}$$

We can therefore easily see that the channel (5.111) is weakly degradable if and only if

$$\frac{\sqrt{2}}{2} \leq |D_S| \leq \mathbb{1}. \tag{5.114}$$

□

5.8 Degradability characterization of fermionic Gaussian channels

In this section we generalize the considerations on fermionic Gaussian channels from the previous sections for the case $\dim\Gamma_S \neq \dim\Gamma_E$. In particular, if the dimension of the environment is at least twice as big as dimension of the system

(i.e. $\mathcal{H}_E = \mathcal{H}_S \otimes \mathcal{H}_S$) then it is always possible to choose a pure environmental state. This leads to the Stinespring representation of quantum channel, in which case the channel properties are environment independent (see discussion in Section 5.2). Having a pure environmental state therefore implies that we can analyze the degradability of the quantum channel (not only the weak degradability as before, where the state of environment was in general mixed, see Section 5.3).

First of all, we need to rewrite the general Gaussian transformation on system and environment. It follows from the discussion in Section 5.6.1 that D_E, the singular value part of the environmental part R_E of the transformation R (see equation (5.69)) can be written in terms of singular value part of the system transformation as

$$D_E = \begin{pmatrix} YD_S & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{5.115}$$

For this case it is easy to see that matrix (5.85) has now the block form

$$\begin{pmatrix} D_S & Y\sqrt{1-D_S^2} & 0 \\ Y\sqrt{1-D_S^2} & D_S & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} =: \begin{pmatrix} D_S & Q_{SE} \\ Q_{ES} & D_E \end{pmatrix}, \tag{5.116}$$

where we introduced the notation $Q_{SE} = \begin{pmatrix} Y\sqrt{1-D_S^2} & 0 \end{pmatrix}$, $Q_{ES} = \begin{pmatrix} Y\sqrt{1-D_S^2} \\ 0 \end{pmatrix} = Q_{SE}^\dagger$ and D_E is given by (5.115).

We can now redo all the steps that have been done for the case $\dim\Gamma_S = \dim\Gamma_E$. In particular, the canonical form of the fermionic Gaussian channel (5.89) is (using the notation of (5.116)):

$$T: \quad \Gamma_S \mapsto D_S\Gamma_S D_S + Q_{SE}\Gamma_E Q_{SE}^\dagger, \tag{5.117}$$

while the generalized complementary channel (5.90) reads

$$T^C: \quad \Gamma_S \mapsto Q_{ES}\Gamma_S Q_{ES}^\dagger + D_E\Gamma_E D_E. \tag{5.118}$$

The degrading map (5.104) then generalizes to

$$W(\Gamma_S) = Q_{ES}D_S^{-1}\Gamma_S D_S^{-1}Q_{ES}^\dagger - Q_{ES}D_S^{-1}Q_{SE}\Gamma_E Q_{SE}^\dagger D_S^{-1}Q_{ES}^\dagger + D_E\Gamma_E D_E. \tag{5.119}$$

The equation (5.119) gives rise to the following generalization of Theorem 5.8

Theorem 5.9. *Let Γ_S be a symbol of the input state and Γ_E the covariance matrices of the environmental state of n-mode fermionic Gaussian channel of the form*

$$T: \quad \Gamma_S \mapsto D_S\Gamma_S D_S + Q_{SE}\Gamma_E Q_{SE}^\dagger. \tag{5.120}$$

Then the channel (5.120) *is degradable if and only if the matrix*

$$M = \begin{pmatrix} -Q_{ES}^\dagger D_S^{-1}\Gamma_E Q_{SE}^\dagger D_S^{-1}R_{ES}^\dagger + D_E\Gamma_E D_E & Q_{ES}D_S^{-1} \\ -(Q_{ES}D_S^{-1})^T & 0 \end{pmatrix} \tag{5.121}$$

meets the condition

$$M^T M \leq \mathbb{1}. \tag{5.122}$$

Let us conclude this section by giving more explicit form of the matrix (5.121). When expressing the blocks Q_{SE} and Q_{ES} and perform the required multiplication we obtain the matrix of the block form:

$$M = \begin{pmatrix} D_E\Gamma_E D_E - \begin{pmatrix} \frac{1-D_S^2}{D_S} & 0 \\ 0 & 0 \end{pmatrix} \Gamma_E \begin{pmatrix} \frac{1-D_S^2}{D_S} & 0 \\ 0 & 0 \end{pmatrix} & Y\frac{\sqrt{1-D_S^2}}{D_S} \\ 0 & \\ \begin{pmatrix} Y\frac{\sqrt{1-D_S^2}}{D_S} & 0 \end{pmatrix} & 0 \end{pmatrix}. \tag{5.123}$$

5.8.1 Example: Degradability of fermionic product channels

Let us again consider the fermionic product channel given by (5.111). In 5.7.2 vacuum has been considered as an environmental state. Taking into account the general form of the Gaussian channel (see (5.89)) we can see, however, that we obtain exactly same channel if we consider the environment to be in pure maximally entangled state if we only couple to the first half of the environment. In that case its covariance matrix will have the form

$$\Gamma_E = \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix}. \tag{5.124}$$

Then we obtain the following expression for the matrix (5.123):

$$M = \begin{pmatrix} 0 & D_S K & \frac{\sqrt{1-D_S^2}}{D_S} \\ -K^T D_S & 0 & 0 \\ \frac{\sqrt{1-D_S^2}}{D_S} & 0 & 0 \end{pmatrix}. \tag{5.125}$$

The condition (5.122) then translates into

$$M^T M = \begin{pmatrix} D_S K K^T D_S + \frac{1-D_S^2}{D_S^2} & 0 & 0 \\ 0 & K^T D_S^2 K & D_S K^T \frac{\sqrt{1-D_S^2}}{D_S} \\ 0 & \frac{\sqrt{1-D_S^2}}{D_S} K D_S & \frac{1-D_S^2}{D_S^2} \end{pmatrix} \leq \mathbb{1}. \tag{5.126}$$

Using the purity of environmental state (in particular $\Gamma_E^T\Gamma_E = \mathbb{1} \Rightarrow KK^T = \mathbb{1}$) the condition for the upper left block can be rewritten

$$-D_S^2 - \frac{1-D_S^2}{D_S^2} + \mathbb{1} \geq 0 \Leftrightarrow (1-D_S^2)^2 \leq 0. \tag{5.127}$$

This condition obviously can not be satisfied for any $0 < D_S < \mathbb{1}$. Having the pure maximally entangled environmental state therefore means that fermionic product channel can never be degradable.

□

If we compare Section 5.7.2 with Section 5.8.1 we see that the weak degradability is indeed the necessary condition for the degradability, but it is not sufficient. The particular choice of highly entangled environment leads to the channel that is neither degradable nor anti-degradable.

In this Chapter we discussed the implications of degradability for the significant simplification of the quantum capacity evaluation. And even though we made arguments that weak degradability is in general not equivalent to degradability to our knowledge there is no known example of a weakly degradable (but not degradable) fermionic channel for which the coherent information would not be additive.

Let us also point out the relation of anti-degradability to the quantum capacity of quantum channel. In particular [20], the anti-degradability of the channel implies zero quantum capacity.

To conlude, in this Chapter we provided an overview of the basic notions of the theory of quantum channels and their capacities. Exploiting the Extremality theorem we have shown how to estimate capacities for the degradable channels. We derived the standard form of fermionic Gaussian channels and provided criteria for their degradability that have been illustrated on the relevant example of fermionic attenuation channel. In the following Chapter we will complete the picture of fermionic Gaussian states in the context of quantum information by introducing the notion of convex Gaussian states and their possible applications in quantum computation theory.

Chapter 6

Beyond Gaussian Setting

While Gaussian states have many desirable properties, one inconvenience is that they do not form a convex set. It is therefore very easy to leave the Gaussian setting. Nonetheless, there was a recent proposal [27], where the convex generalization of fermionic Gaussian states was introduced and some implication for quantum computation have been shown.

In this Chapter we would like introduce the convex hull of the set of fermionic Gaussian states and give some possible applications of this framework. In order to fully appreciate the computational power of fermionic Gaussian mixtures some introductory knowledge on possibility of turning fermionic systems into universal quantum computer will be needed. For that purpose we will also give a short review on classical simulatability of fermionic linear optics.

6.1 Fermionic linear optics and classical simulatability

As emphasized many times, encouraged by recent experimental progress with fermionic systems we decided to answer some questions about how many of the attractive results known from the bosonic settings can be translated to the fermionic one. In previous chapters we have seen how to describe states and various transformations. We saw that Gaussian fermionic states exists and they also have important extremality property same as bosonic ones. There is however one more important point to be addressed and that is computional power of fermionic systems.

One of the first discoveries that drew the attention to the quantum information and computation was the possible exponential speedup that quantum systems offer for some tasks compared to classical ones (see cf. [75]). There is actually an overlap of this issue with the topics regarding the Gaussian framework.

First, let us recall that by fermionic linear optics (FLO) we understand the class of gauge-invariant unitaries that map Gaussian states on Gaussian states. These have been precisely characterized in Chapter 2. Let us also note that there is a small incoherence in the community regarding the use of the term 'fermionic linear optics'. While some authors use FLO to denote Gaussian unitaries plus

single mode measurements (see for instance [15]), the others, same as we do, by FLO mean Gaussian unitaries exclusively (see [32]). So whenever we will talk about unitaries and measurements we will emphasize that explicitly.

The bosonic counterpart of this class of unitaries consists of linear optical elements (like beamsplitters, phase shifters etc.). For bosons there is a proof [61] that linear optics plus single mode measurement (photon detection) are enough to reach an exponential speedup.[1]

This is of course something that is not ensured when the statistics of the system is changed. In fact, there is a proof of an exactly opposite statement for fermions, so-called no-go theorem [79] stating that exponential speedup of quantum over classical algorithms cannot be reached making use only FLO and single mode measurements only.

It would make fermionic systems much less attractive if they would not have provided any of the quantum computation benefits. Luckily it has been shown by Beenaker at al. [6] that we can enter the regime where universal quantum computation is possible by adding the two-mode measurement to our tools. There is very illustrative analysis of the simulatability problem in terms of Slater determinants given by Terhal and DiVincezo in [32]. We will follow some of their arguments to see why the universal quantum computation is not possible in the fermionic Gaussian setting.

Knowing how the change of statistics is influencing the symmetry of the wavefunction we can argue that the fermionic simulatability statement is not that quite surprising. While bosonic function is fully symmetric, which means that we can write any wavefunction of the many-body system as a permanent of the wavefunctions of the respective particles, fermionic wavefunction has to be anti-symmetric and that correspond to the determinant (so-called Slater determinant, see [59]).

And while for computation of the determinant there is simple efficient algorithm, there is no such thing to calculate permanent. That alone suggests an existence of significant difference in classical simulatability. Before starting a Slater determinant characterization of fermionic states let us present a simple example that illustrates how anti-symmetry described above translates into our prefered second-quantization picture.

6.1.1 Example: second quantization of fermionic wavefunction and entanglement

Let us consider the system with two accessible orbitals a_1, a_2, in position representation we will denote orthonormal basis single particle wavefunctions $\{\Phi_{a_1}(r_1)$, $\Phi_{a_2}(r_2)\}$. As the first example we choose the situation that we put one particle into orbital a_1 and one particle into orbital a_2. Then the antisymmetrized wavefunction reads

$$\Psi(r_1, r_2) = \frac{1}{\sqrt{2}}(\Phi_{a_1}(r_1)\Phi_{a_2}(r_2) - \Phi_{a_1}(r_2)\Phi_{a_2}(r_1)). \tag{6.1}$$

[1] for bosons, the photon detection is not Gaussian operation.

This is what we will call Slater determinant or determinantal wavefunction and what straightforwardly generalizes to the higher dimensions. The reason for the name 'determinant' is rather obvious, the expression (6.1) can be seen as a determinant of the matrix:

$$\begin{pmatrix} \Phi_{a_1}(r_1) & \Phi_{a_1}(r_2) \\ \Phi_{a_2}(r_1) & \Phi_{a_2}(r_2). \end{pmatrix} \tag{6.2}$$

In the second quantization language we can rewrite equation (6.1) as

$$a_1^\dagger a_2^\dagger \left|0\right\rangle, \tag{6.3}$$

because here anti-symmetry is built in the creation and annihilation operators.

Next question is then whether we can write all states as some variation of the expression (6.1) resp. (6.3). This is obviously in general not true. Let us consider the system with more modes available, say four with respective creation operators: $a_1^\dagger, a_2^\dagger, a_3^\dagger, a_4^\dagger$. Then we can for example think of the state $\left|\Psi\right\rangle = a_1^\dagger a_2^\dagger \left|0\right\rangle + a_3^\dagger a_4^\dagger \left|0\right\rangle$, which can not be written as single Slater determinant [62].

In fact, the number of Slater determinants needed to describe given state is one of the entanglement measures, so-called Slater rank (see [35]).

□

Let us again start off with the one-particle Hilbert space $\mathcal{H}$ of dimension d with basis $\{\left|i\right\rangle\}, 1 \leq i \leq d$. Then the state with the fermion in the i-th orbital will be expressed as $a_i^\dagger \left|0\right\rangle$. If we wish to describe the state of more than one fermion, say $n \leq d$ then one example would be

$$\prod_{i=1}^{n} a_i^\dagger \left|0\right\rangle. \tag{6.4}$$

In the discussion in 6.1.1 we have seen that not every state can be written in this form. More concretely, the states expressible as (6.4) as those that can be written as a single Slater determinant. What we will do next is to show that when starting off Slater rank-one state we cannot reach the states with the higher Slater rank only by means of fermionic linear optics and single mode measurements.

First as discussed in the chapter 2 FLO transformation on the non-interacting fermions (aka unitary evolution governed by non-interacting hamiltonian) does not increase Slater rank. The we will argue the same for the von Neumann single mode measurement. We will also comment on increase of Slater rank in case of two- and multi-mode measurement.

Adjoint action of FLO alone on the product (Slater rank-one) state is an immediate result of what has been discussed in the chapter (2). We can write the adjoint action of allowed unitaries $U(t)$, where t denotes time-dependence, as

$$U(t) a_i^\dagger U^\dagger(t) = \sum_m V_{im}(t) a_m^\dagger := a_i^\dagger(t), \tag{6.5}$$

where by $a_i^\dagger(t)$ we mean the creation operator for the i-th mode if the time t-evolved system. So we can see the time evolution merely like the rotation of basis chosen to fix the creation and annihilation operators of the system. Applying this to the product state (6.4), unitarity of U and the fact that $U|0\rangle = |0\rangle$ we obtain

$$U(t)\prod_{i=1}^{n} a_i^\dagger U^\dagger(t)\,|0\rangle = \prod_{i}^{n} a_i^\dagger(t)\,|0\rangle\,. \tag{6.6}$$

Next item on our list were single mode von Neumann measurement. Let us choose one of the accessible modes, say $|j\rangle$. When measuring occupancy we have two options, either one particle or none. Therefore the von Neumann measurement will have two elements being

$$P_1 = a_j^\dagger a_j \tag{6.7}$$

and

$$P_0 = \mathbb{1} - a_j^\dagger a_j = a_j a_j^\dagger. \tag{6.8}$$

What we want to see now is that when applying (6.7) resp. (6.8) on (6.4) Slater rank will not be increased. Here we will inspect in detail only the case of the projector (6.7), the situation for (6.8) is analogous and is calculated in detail in [32].

The strategy of the proof is following. We started from the accessible orbitals $\{|i\rangle\}$, performed a sequence of FLO operations (without changing Slater rank) to obtain the set of time evolved orbitals $\{|i(t)\rangle\}$, where each member of the set is a linear combination of original orbitals $|i\rangle$. Now we will write on orbital $|j\rangle$ of the interest as the linear combination of the states

$$\{|\mathrm{in}\rangle\,, |\mathrm{out}\rangle\}, \quad |\mathrm{in}\rangle \in \mathrm{Span}(\{|i(t)\rangle\}), \quad |\mathrm{out}\rangle \in \mathrm{Ker}(\{|i(t)\rangle\}) \tag{6.9}$$

as

$$|j\rangle = \alpha\,|\mathrm{in}\rangle + \beta\,|\mathrm{out}\rangle\,, \tag{6.10}$$

where α, β are real, non-negative (this is just a matter of the choice of the phase) with the property $\alpha^2 + \beta^2 = 1$ (normalization).

Introducing the new orbitals $\{|\mu\rangle\}$ such that we can write $\mathrm{Span}(\{|i(t)\rangle\})$ as

$$\mathrm{Span}(\{|i(t)\rangle\}) = \mathrm{Span}(|\mathrm{in}\rangle\,, \{|\mu\rangle\}). \tag{6.11}$$

we can write the time-evolved product state (6.6) as [32]

$$\prod_{i}^{n} a_i^\dagger(t)\,|0\rangle = a_{\mathrm{in}}^\dagger \prod_{\mu} a_\mu^\dagger\,|0\rangle\,. \tag{6.12}$$

That allows us to calculate

$$P_1\left(\prod_i^n a_i^\dagger(t)\left|0\right\rangle\right) = a_j^\dagger a_j a_{\text{in}}^\dagger \prod_\mu a_\mu^\dagger \left|0\right\rangle = \\ = a_j^\dagger(\alpha a_{\text{in}} + \beta a_{\text{out}}) a_{\text{in}}^\dagger \prod_\mu a_\mu^\dagger \left|0\right\rangle = \\ = \alpha a_k^\dagger \prod_\mu a_\mu^\dagger \left|0\right\rangle .$$

So, up to normalization we stayed in Slater rank-one settings.

Now we turn to two-mode measurements scenario. Let us denote the modes of interests $|j\rangle$ and $|k\rangle$. Here the von Neumann measurements consists of three elements: both modes are occupied, none of them is,or, there is a particle in only one of the modes, i.e.:

$$P_2 = a_j^\dagger a_j a_k^\dagger a_k, \tag{6.13}$$

$$P_0 = a_j a_j^\dagger a_k a_k^\dagger, \tag{6.14}$$

$$P_1 = a_j a_j^\dagger a_k^\dagger a_k + a_j^\dagger a_j a_k a_k^\dagger. \tag{6.15}$$

So we see that in that case of projectors (6.14) and (6.13) we only do the procedure described for the case of the single-mode von Neumann measurement twice in the row which is why the Slater number can not increase.

There is however proof in [32] that Slater rank can increase in case of (6.15). In particular, it holds that after applying projector (6.15) on states of the form (6.4) the Slater rank of the outcome equals 2. It is also conjectured in [32] that Slater rank grows exponentially when applying (6.15) repeatedly which is agreement with the proof of Beenakker et al. [6] that predicts universal quantum computation if we add two-mode measurements to the FLO and single-mode measurements. Given the progress in quantum dots systems, where the non-demolition two-mode measurement has been successfully performed [42] this computational scheme could be realizable in the solid state systems.

To conclude, in this section we commented on computational power of fermionic linear optics in terms of Slater determinants.

The knowledge of classical simulatability of Gaussian states can be useful for decision whether the other computational schemes are classically simulatable or not. For that purpose so-called convex-Gaussian states have been introduced and applied on the particular problem of topological quantum computation. In the next section we will comment on both this framework and its application.

6.2 Beyond Gaussian states

Using the already introduced standard form of a Gaussian state

$$\rho = \frac{1}{2^n}\prod_k (\mathbb{1} + i\lambda_k c_{2k-1} c_{2k}) \tag{6.16}$$

it can be shown that on the one hand every Gaussian state can be regarded as a convex mixture of Gaussian states. Let us rewrite the coefficients λ_k's as $2p_k - 1$ with $p_k \in [0, 1]$, i.e. the state will have the form

$$\rho = \frac{1}{2^n} \prod_k (\mathbb{1} + i(2p_k - 1)c_{2k-1}c_{2k}) = \prod_k [p_k(\mathbb{1} + ic_{2k-1}c_{2k}) + (1 - p_k)(I - ic_{2k-1}c_{2k})]. \tag{6.17}$$

On the other hand not every convex mixture of Gaussian states is Gaussian (see for instance [2]). In [27] de Melo et al. asked what we can conlude about the wider class of states consisting if convex hull of Gaussian states. In particular, the definition of convex-Gaussian state is following.

Considering the Clifford algebra $\mathcal{C}_{2n}$ an even density matrix $\rho \in \mathcal{C}_{2n}$ is convex-Gaussian if and only iff it can be written as a convex combination of pure Gaussian states in $\mathcal{C}_{2n}$, i.e.

$$\rho = \sum_i p_i \sigma_i, \tag{6.18}$$

where σ_i's are pure Gaussian states and $p_i \geq 0$ and $\sum_i p_i = 1$.

Then making use of the following properties of covariance matrix: (that already have been discussed in previous chapters)

- covariance matrix has eigenvalues $\pm i\lambda_k$, where λ_k are Williamson eigenvalues
- $\Gamma^T \Gamma \leq \mathbb{1}$
- $\Gamma^T \Gamma = \mathbb{1}$ if and only if it is a covariance matrix of a pure Gaussian state

the following theorem can be shown:

Theorem 6.1. *For every even state ρ there exist a sufficiently small $\epsilon \geq 0$such that the state $\rho_\epsilon = \epsilon\rho + (1 - \epsilon)\frac{\mathbb{1}}{2^n}$ is convex-Gaussian.*

Knowing this we would like to ask an opposite question, whether for a given state there exists Gaussian decomposition. This question can be answered using a criterion by Bravyi [15], that we have already cited in chapter 3: an even state $\rho \in \mathcal{C}_{2n}$ is Gaussian if and only if

$$[\Lambda, \rho \otimes \rho] = 0 \tag{6.19}$$

for

$$\Lambda = \sum_i c_i \otimes c_i, \tag{6.20}$$

where c_i stands for Majorana operators (2.8).

Let us derive a necessary and sufficient criterion for the state to be pure Gaussian.

First we claim that (6.20) is invariant under FLO operators of the form $U \otimes U$. To show that first note that

$$\tilde{c_i} = U c_i U^\dagger = \sum_i O_{ij} c_j,$$

where $O := O_{ij}$ is an orthogonal matrix. Then we can write

$$\begin{aligned} U \otimes U \left(\sum_i c_i \otimes c_i \right) (U^\dagger \otimes U^\dagger) &= \sum_{ijk} O_{ij} c_j \otimes O_{ik} c_k = \\ &= \sum_{jk} \left(\sum_i (O_{ji})^T O_{ik} c_j \otimes c_k \right) = \\ &= \sum_j c_j \otimes c_j. \end{aligned} \tag{6.21}$$

Using this we can rewrite the trace norm $||.||_1$ of adjoint action of (6.20) on the state $\rho \otimes \rho$ as follows

$$||\Lambda\rho \otimes \rho\Lambda||_1 = \sum_{ij} \mathrm{Tr}(c_i \otimes c_i \rho \otimes \rho c_j \otimes c_j) = \tag{6.22}$$

$$= \sum_{ij} [\mathrm{Tr}(c_i c_j \rho)]^2 = \tag{6.23}$$

$$= \sum_i [\mathrm{Tr}(c_i^2 \rho)]^2 - \sum_{i \neq j} [\mathrm{Tr}(c_i c_j \rho)]^2 = \tag{6.24}$$

$$= 2n - \sum (\Gamma_{ij})^2 = 2n - \mathrm{Tr}\Gamma^T \Gamma. \tag{6.25}$$

In (6.22) we have used the fact that we work with positive operators only, therefore absolute value in the definition of the trace can be omitted and we can pull the sum out of the trace, (6.23) is application of the property $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}A \cdot \mathrm{Tr}B$. For review and proofs of these see for instance [70]. In (6.25) we merely applied that $c_i^2 = 1, \forall i$ and the definition of covariance matrix (2.22).

In addition to that we know that for a pure Gaussian state $M^T M = \mathbb{1}$, i.e. for a pure state RHS of (6.25) equals 0. Let us recall the criterion of the purity of the state in terms of density matrix [68]. State ρ is pure if and only if its density matrix meets the condition

$$\rho^2 = \rho. \tag{6.26}$$

Then we can write

$$||\Lambda\rho \otimes \rho\Lambda||_1 = ||\Lambda(\rho \otimes \rho)^2 \Lambda||_1 = ||\Lambda(\rho \otimes \rho)[\Lambda(\rho \otimes \rho)]^\dagger||_1 = ||\Lambda(\rho \otimes \rho)||_2. \tag{6.27}$$

In other words if $||\Lambda\rho \otimes \rho\Lambda||_1 = 0$, then $||\Lambda(\rho \otimes \rho)||_2 = 0$ which holds if and only if $\Lambda(\rho \otimes \rho) = 0$. Again, for details see [70].

So we can conlude that even state on Clifford algebra $\mathcal{C}_{2n}$ is pure Gaussian state if and only if

$$\Lambda\rho \otimes \rho = 0. \tag{6.28}$$

It is proven in [27] that the kernel on Λ is contained in the symmetric subspace of $\mathbb{C}^n \otimes \mathbb{C}^n$ We have then that all the product states of two copies of the pure Gaussian state are in the kernel of the operator (6.20), so the kernel of Λ is spanned by symmetric states of the form $|\Psi, \Psi\rangle$, where Ψ is Gaussian state.

Inspired by the separability criteria based on existence of infinite symmetric extensions (see [33], [34]) the authors of [27] formulate the test on the state having a symmetric extension that is Gaussian in the form that can be rewritten as semi-definite program [27]:

Program. Input: *$\rho \in \mathcal{C}^{2m}$ and an integer $n \geq 2$*

Body: *Is there a $\rho_{ext} \in \mathcal{C}^{2m}_{\otimes n}$ sth.* $\mathrm{Tr}_{2,\ldots,n}\rho_{ext} = \rho$, $\mathrm{Tr}\rho_{ext} = 1$, $\Lambda^{k,l}\rho_{ext} = 0, \forall k \neq l$, $\rho_{ext} \geq 0$

Output: *yes (provide ρ_{ext}) or no*

If this program provide an answer 'yes' we say that the state has Gaussian-symmetric extension. Next step is to prove that only convex-Gaussian states have Gaussian-symmetric extension (by which me mean, that there is no n for which would the program above provide answer 'no', reproducing the terminology of [33], [34], we say that our criterion is complete).

Theorem 6.2. *An even state ρ has Gaussian symmetric extension to an arbitrary number of parties if and only if ρ is convex Gaussian.*

The proof of this statement is a simple application of quantum de Finetti theorem [22] and can be found again in [27].

So we defined convex-Gaussian states and introduce the criteria how to test whether the state under consideration is in this class or not. Since the main purpose of the whole chapter was analyze computation power of fermions with respect to the Gaussian settings we wish to conclude by pointing out one remarkable computational application of convex-Gaussian states.

There is a result by Bravyi [16] analyzing how to turn topological quantum computation (TQC) (see for instance [37]) into universal quantum computation. This indeed can be done if TQC is supplemented by non-topological operations that can be very noisy. The amount of this artificially added noise is connected to the classical simulatability of the system. Bravyi established this model using two ancillary states two-mode $|a_4\rangle \in \mathcal{C}_4$ and four-mode $|a_8\rangle \in \mathcal{C}_8$. Now, since the non-topological part of the operation is not perferct we only prepare some mixed states, lets call them ρ_4, ρ_8 that will resemble $|a_4\rangle$ and $|a_8\rangle$. We will define a measure for the precision of this approximation using the fidelity $F = \langle a_4| \rho_4 |a_4\rangle$

$$\epsilon_4 = 1 - \langle a_4|\rho_4|a_4\rangle \tag{6.29}$$

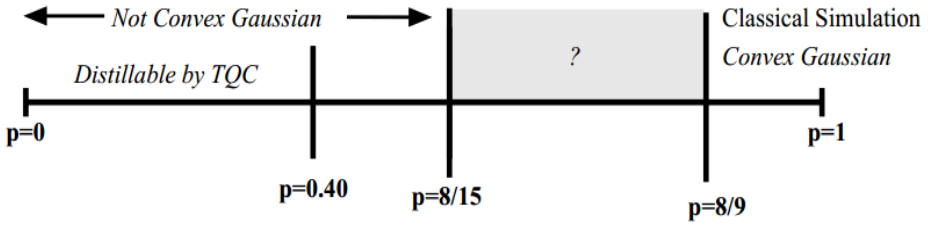

Figure 6.1: The results of convex-Gaussian analysis of the noisy ancilla $|a_8\rangle$ for universal quantum computation given in [27].

and

$$\epsilon_8 = 1 - \langle a_8|\rho_8|a_8\rangle. \tag{6.30}$$

Bravyi then established that the ideal states $|a_4\rangle$ and $|a_8\rangle$ can be distilled from noisy ones provided that:
1. all topological operations are perfect
2. $\epsilon_4 < 0.14$
3. $\epsilon_8 < 0.38$.

It above three conditions are met then any quantum computation can be simulated by topological means exclusively. So roughly speaking, indeed there is some noise needed in order to leave a classical simulatebility of TQC alone, but there is a thereshold (in case of [16] characterized by ϵ_4 and ϵ_8) above which the universal quantum computation is not possible again.

In [27] this threshold is characterized and extended using convex-Gaussian framework. We can see the threshold for universal quantum computation problem from the other side: we can ask how much noise we need to add in arder to enter the regime where the ancillary states have Gaussian decomposition (i.e. clasically simulatable situation). And since we have semi-definite programm at hand to test convex-Gaussianity we can apply it on Bravyi's ancillas undergoing depolarization. In [27] the allowed noise on ϵ_8 has been extended to $\epsilon_8 = \frac{8}{15}$ and in addition to that it has been shown that the Gaussian decomposition exists for $\epsilon_8 \geq \frac{8}{9}$, which leaves so far unresolved grey zone for $\frac{8}{15} \leq \epsilon_8 \leq \frac{8}{9}$ (see Figure 1).

Let us conclude the discussion of convex Gaussian states by a short remark on the special case, when the convex combination actually is Gaussian state. Even though it is little out of the scope of this chapter it can be seen as a nice connection to the characterization of Gaussian states done in Chapter 2. There is a condition [89] connecting the Gaussianity of the mixture to the rank of the difference of the symbols making up the convex combination. In particular, for the Gaussian states ω_{Λ_1}, ω_{Λ_2} fully characterized by the symbols Λ_1, Λ_2 the convex mixture $\lambda\omega_{\Lambda_1} + (1-\lambda)\omega_{\Lambda_2}$ is a Gaussian state with the symbol $\lambda\Lambda_1 + (1-\lambda)\Lambda_2$ if and only of $\Lambda_1 - \Lambda_2$ has rank either 0 or 1.

In this chapter we adressed the computation using fermionic maps and states. First we connected the classical simulatability with the Slater rank of the state

and then we analyzed the change of this number under fermionic linear optics, one- and two-mode measurements. The conclusion of this part of our analysis was that while the FLO with single-mode measurements is clasically simulatable, after adding two-mode measurements in the picture we enter the regime of universal quantum computation. In the second part of this chapter, we established convex generalization of the Gaussian states and gave a review of the result that uses this technique to improve noise threshold for topological quantum computation supported by noisy ancillas.

Chapter 7

Applications Outlook

Let us give a short outlook regarding the physical systems in which we can make use of the framework introduced in this work. Given the recent experimental progress in engineering and control of solid state systems there are some promising candidates for the physical realization of the framework introduced in this work.

The Gaussian settings described here is well suited for describing quantum information transmission (i.e. quantum channels) rather than quantum information processing. While in quantum information processing the non-Gaussian operation are necessary, in quantum information transmission we aim for the channel with the minimal information losses, ideally the perfect one. The Gaussian models are therefore much more likely to be of relevance within quantum channels settings.

On the other hand, in spite of the radical simplification of the Gaussian approximation, Gaussian states still have some nontrivial topological properties, that can be used for storing and manipulating information.

Let us start with possible implementation of the quantum channel. In this case we can use the Gaussian description introduced in this Thesis to model the noisy information transmission.

First candidate for quantum information processing with fermions are the cold atoms [38]. Fermions can be cooled down and trapped in optical lattices [54] and their transport through the lattice can be then achieved by manipulating the potential [72]. The transport is realized as a hopping on the site level, which gives rise to a Gaussian channel.

Another promising system for storing and manipulating quantum information are quantum dots, artificially structured systems confined in three spatial dimension that can be filled with electrons and coupled via tunnel barriers [64]. There are various strategies how to implement the quantum channels in these structures [41]. For instance building quantum dots arrays [18], using the edge currents in quantum Hall systems as information carriers in between the modes [44], [67] or transporting electrons via surface acoustic wave [45].

Regarding the non-trivial topological properties of states we can think of the Majorana fermions, that we used simply as convenient representation for some purposes, as of actual particles. Note that, because of the hermiticity, Majorana

fermion is its own anti-particle [83]. In the fermionic anti-commutation setting the exchange of particles is a non-trivial operation. This fact can be explored in connection to topological quantum computation [58]. There is an extensive search for Majorana fermions in condensed matter systems (see for example [1]). Even though some signs of Majorana modes have been observed (for instance [36]), their existence is still unconfirmed.

Chapter 8

Conclusion

In this Thesis we studied fermionic Gaussian states in the context of quantum information. We gave an overview of the methods for their description and provided the translational map between different approaches. We derived the standard form of fermionic Gaussian maps and channels. We provided the degradability criteria for fermionic Gaussian channels and discussed their impact on the evaluation of quantum capacity of quantum channels. The general degradability criteria have been used to obtain full degradability classification of fermionic product channels.

This work could be continued by the more detailed analysis of degradability. The conditions provided in this Thesis are completely general. It might be however possible to simplify them for the particular regimes of parameters that have been used for the channel characterization and to obtain a full degradability classification, e.g., for channels on small systems.

Concerning the physical application of these results, an analysis of recently implemented fermionic channels in solid state systems would be interesting to see how close these channels are to the Gaussian regime.

We analyzed teleportation protocol for fermions. This is the first step for exploiting the teleportation channel in order to find the lower bounds on the channel capacities.

Using Jordan-Wigner transformation we can map fermionic Gaussian channels on qubit channels and see whether we can recover new class of qubit channels. Within qubit channels we could also further explore the generalized Gaussian states since the parity superselection rule no longer applies.

In general, Gaussian approximation is the first step for understanding the many-body systems. Having tools of convex Gaussian states available could lead to the analysis of non-Gaussian regime, while the complexity is kept at sufficiently simple level.

Appendix A

Teleportation with fermions

A.1 Introduction

In this Appendix we discuss one example of a quantum channel of great importance in quantum information theory. It is the so-called teleportation channel proposed by Bennett [8]. Roughly speaking it is the protocol that, using non-locality of quantum mechanics, allows to reconstruct an unknown quantum state at a spatially separated location by sending classical information only.

The quantum teleportation protocol has drawn a lot of attention not only because of its, from classical point of view, counterintuitiveness. It is also one of the possible tools for universal quantum computation [40] that provides a reduction in resource requirements. This so-called teleportation-based quantum computation has broad conceptual implications as well. For instance, an equivalence with cluster-state-based quantum computation model (see for example [55]) has been shown [81].

We want to discuss the teleportation channel here because it constitutes a very important and illustrative example of the situation when the features of fermionic statistics, namely the parity superselection rule, represent an obstacle of the proposal that works flawlessly for bosonic particles.

In this Chapter we give an overview of the original Bennett's protocol and comment on its recent generalizations. Then we explain the difficulties coming with the superselection rule and show a method how to overcome this issue. We then conclude by the formulation of the open questions.

A.2 Quantum teleportation protocol

Let us consider as many times before a sender, Alice, and a receiver, Bob. Let us assume that they share a maximally entangled quantum state. A question arises whether we can use this as a quantum channel for sending quantum information. The answer, provided that we can perform certain measurements and we have classical communication channel available, is yes [8].

As we said, Alice and Bob share maximally entangled state of two qubits. It turns out that a good choice of basis for quantum information transmission problem is the so-called Bell basis of maximally entangled two qubit states:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{A.1}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{A.2}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{A.3}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{A.4}$$

The states (A.1)-(A.4) are related to each other via local unitary transformations. In particular, starting, for example, from the so-called singlet state Ψ^-, we get the other three Bell pairs as follows

$$\sigma_x \otimes \mathbb{1} \, |\Psi^-\rangle = - \, |\Phi^-\rangle \, , \tag{A.5}$$

$$\sigma_y \otimes \mathbb{1} \, |\Psi^-\rangle = i \, |\Phi^+\rangle \, , \tag{A.6}$$

$$\sigma_z \otimes \mathbb{1} \, |\Psi^-\rangle = |\Psi^+\rangle \, , \tag{A.7}$$

where by $\sigma_x, \sigma_y, \sigma_z$ we denote the Pauli operators.

Let us now assume that Alice has a qubit in (an unknown) quantum state, say

$$|\varphi\rangle = \alpha \, |0\rangle + \beta \, |1\rangle \, , \quad \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \tag{A.8}$$

and that the two-qubit entangled state that Alice and Bob share is the singlet state (A.1). This means that the overall state of three qubits under consideration is

$$|\varphi\rangle_a \otimes |\Psi^+\rangle_{AB} = (\alpha \, |0\rangle_a + \beta \, |1\rangle_a)(\frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})), \tag{A.9}$$

where we introduced the subscript a for the qubit to be transmitted, A for Alice's, and B for Bob's half of the singlet, respectively. Using the basis (A.1)-(A.4) we however see that we can rewrite the RHS of equation (A.9) as

$$\begin{aligned} |\varphi\rangle_a \otimes |\Psi^+\rangle_{AB} = \frac{1}{2}[&|\Psi^-\rangle_{aA} \, (-\alpha \, |0\rangle_B - \beta \, |1\rangle_B) + \\ &+ |\Psi^+\rangle_{aA} \, (-\alpha \, |0\rangle_B + \beta \, |1\rangle_B) + \\ &+ |\Phi^-\rangle_{aA} \, (\alpha \, |0\rangle_B + \beta \, |1\rangle_B) + \\ &+ |\Phi^+\rangle_{aA} \, (\alpha \, |0\rangle_B - \beta \, |1\rangle_B)]. \end{aligned} \tag{A.10}$$

If Alice now measures the qubits a, A in the Bell basis, she obtains an outcome corresponding to one of the states (A.1)-(A.4) with equal probability. One can

The possible events in a teleportation protocol.

Alice's Bell meas.	Inferred spin properties			State of Bob's qubit	Required unitary transf.
	$\hat{\sigma}_x \otimes \hat{\sigma}_x$	$\hat{\sigma}_y \otimes \hat{\sigma}_y$	$\hat{\sigma}_z \otimes \hat{\sigma}_z$		
$\|\Psi^-\rangle_{aA}$	-1	-1	-1	$\|\psi\rangle$	$\hat{\mathrm{I}}$
$\|\Psi^+\rangle_{aA}$	$+1$	$+1$	-1	$\hat{\sigma}_z\|\psi\rangle$	$\hat{\sigma}_z$
$\|\Phi^-\rangle_{aA}$	-1	$+1$	$+1$	$\hat{\sigma}_x\|\psi\rangle$	$\hat{\sigma}_x$
$\|\Phi^+\rangle_{aA}$	$+1$	-1	$+1$	$\hat{\sigma}_y\|\psi\rangle$	$\hat{\sigma}_y$

Figure A.1: The overview of all possible events in the teleportation protocol [5].

easily see that this measurement will leave the qubit B on Bob's side in the state that corresponds to (A.8) up to local unitary Pauli rotation which is completely determined by measurement outcome (see Figure A.1 for an overview of the correction operations for a given Bell measurement outcome). Having performed the measurement Alice sends its outcome as two-bit classical message. Bob then performs respective local unitary after which he with certainty posses the state that had been sent.

This explanation is simply to illustrate the principle of the protocol. For a translation of the teleportation scheme into the general language of representations of quantum channels (as it has been described in the Chapter 5) see [82]. In fact, [82] gives a general framework that allows to describe all conceivable teleportation as well as dense coding (see [9]) protocols on the same footing.

A.3 Quantum teleportation for fermions

Now when we understand the protocol we can consider to employ fermions to carry out quantum teleportation. As emphasized at the beginning of this Chapter, there is, however, the parity superselection rule to be taken into account. Its implication for the protocol described above is that when applying postprocessing unitaries we can create or destroy the particles in the system only in pairs.

When we take a look at the Figure A.1 we can immediately come up with an example of a violation the parity superselection. Using the very definition of the operator σ_x we get:

$$\sigma_x |0\rangle = |1\rangle , \quad \sigma_x |1\rangle = |0\rangle . \tag{A.11}$$

This operation violates the parity superselection rule (as well as σ_y), which is why

whenever Alice obtains the measurement result corresponding to the correction unitaries σ_x or σ_y, Bob is prevented by parity superselection rule to obtain correct output state.

The way how to overcome this difficulty proposed by Morgenshtern et al. [66] is to add locally an ancillary fermion mode in the vacuum state. Without changing the parity we can then safely perform the following local transformations

$$\begin{aligned} |00\rangle &\mapsto |11\rangle \\ |11\rangle &\mapsto |00\rangle \\ |01\rangle &\mapsto |10\rangle \\ |10\rangle &\mapsto |01\rangle . \end{aligned} \tag{A.12}$$

Using these we can transform (again without parity superselection rule violation)

$$|0\rangle |\Psi^{\pm}\rangle \mapsto |1\rangle |\Phi^{\pm}\rangle \tag{A.13}$$

and

$$|0\rangle |\Phi^{\pm}\rangle \mapsto |1\rangle |\Psi^{\pm}\rangle . \tag{A.14}$$

In other words, the presence of an ancillary mode allows us to switch between different (two-mode) parity subspaces without changing the parity of the overall system.

Let us note that this idea of adding an additional ancillary mode has been extended to many body system by generalization of [81]. As pointed out at the beginning of this Chapter, teleportation and measurement-based quantum computation have been proven to be equivalent. Measurement based quantum computation for fermions is then again analogous to resolving the problem of fermionic teleportation. This has been done in [21] by combining arguments of [81] and [66]. In particular, the measurement-based quantum computation is again seen in the valence bond picture, however the number of bonds used to describe is increased via the procedure of Morgenshtern at al. described above for a single mode. Note that all the operations needed in this protocol are Gaussian.

This procedure described above in priciple resolves the issue of fermionic teleportation. It is, however, experimentally challenging, since it requires interactions. In the following section we will describe another strategy that might lead to the experimentally less demanding approach.

A.4 Port-based teleportation

In the previous sections we have understood that the problem of accommodating the quantum teleporation protocol into the fermionic world relies on performing the correction unitaries that might not preserve parity of the state. This issue has been resolved by adding the ancillary mode in the system, which, however, leads to the complicated experimental scheme.

There is a recent proposal called port-based teleportation (see [52], [53]) that differs from Bennett's protocol by the measurement choice. The measurement being done is a more complicated general POVM (see Definition A.1), which comes with advantage of no correction unitaries whatsoever in this protocol. As for every POVM protocol there is a finite probability that the protocol give us indecisive answer. Probability of such answer, however, goes to zero as the number of POVM elements goes to infinity.

Before explaining the protocol in detail let us recall the definition of POVM using the following simple example from [68].

A.4.1 Example: POVM measurement

Definition A.1. *POVM (stands for positive operator valued measures) is defined as a set of positive operators* $\{E_i\}$ *such that* $\sum_i E_i = \mathbb{1}$.

Note that compared to the standard von Neumann measurement the orthogonality condition is missing. Let us consider the situation that we would like to distinguish states $|0\rangle$ and $|0\rangle + |1\rangle$. Now with the von Neumann measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ when projecting on $|1\rangle$ we obtain the right answer, while when projecting on $|0\rangle$ we learn nothing.

Let us now define more general POVM as follows:

$$E_1 = \lambda |1\rangle\langle 1| \tag{A.15}$$
$$E_2 = \lambda(|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \tag{A.16}$$
$$E_3 = \mathbb{1} - E_1 - E_2. \tag{A.17}$$

Now we see that the results corresponding to E_1, E_2 imply that we have state $|0\rangle$, $|0\rangle + |1\rangle$ respectively. The outcome corresponding to E_3 is ambiguous result. In case of outcome 3 we simply repeat the procedure until we obtain 1 or 2.

We can cocnlude then that POVM allow conclusive decision even with single measurement being done (for the price of a possibly ambiguous outcome).

□

The POVM principle has been successfully applied to the teleportation protocol in [52], [53]. The main principle is illustrated in the Figure A.2. The protocol is based on the fact that Alice and Bob do not share just a single Bell pair, but N of them. In addition to that, Alice has the qubit C to be teleported. She performs POVM on the qubits $C, A_1, \ldots, A_N$ and tells Bob the result correspong to the port B_i via the classical communication channel. The procedure does not require any post-processing on Bob's side, but there is a finite probability that the POVM will return an indecisive outcome and the procedure needs to be repeated.

There is a derivation of the optimal POVM that provides highest probability for any number of Bell pairs in [53]. Even though the port-based protocol provides reasonable success probabilities and is post-processing free, it is undisputably

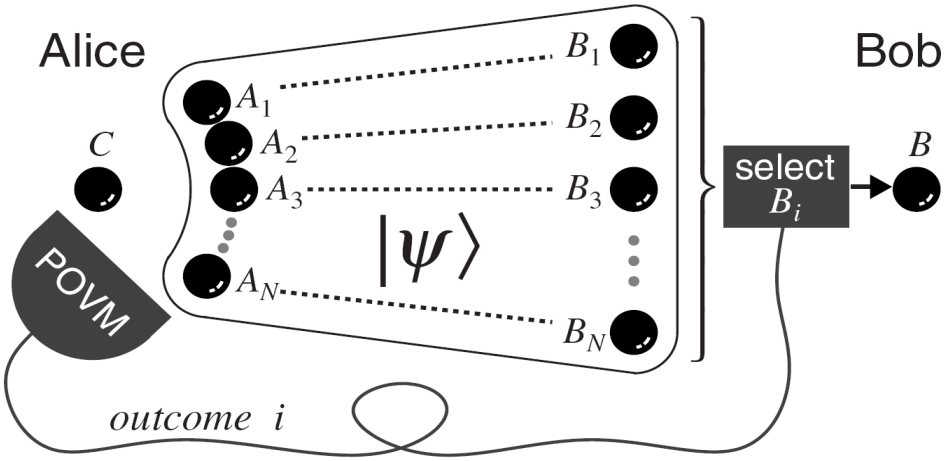

Figure A.2: Port-based teleportation setting [53]

demanding in terms of highly entangled resource state and very complicated measurement. Nonetheless, there is a procedure called entanglement recycling [78] that uses the entanglement left in the resource state for the next round of teleportation, which reduces the cost of entangled pair per successfully teleported qubit.

The absence of post-processing is the clearly the reason why this protocol could be of interest with respect to quantum information processing with fermions. It is however an open question how to translate this protocol into fermionic language.

To conclude, in this Chapter we chose the well-known teleportation protocol to illustrate the implications of parity superselection rule. We explained why parity conservation prevents certain transformations crucial for the algorithm. We commented on an improvement of the setting such that the teleporation could be performed. We also described port-based teleporation, another promising candidate on resolving fermionic teleportation protocol.

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Declaration

I, Eliška Greplová, hereby declare that I have written this thesis independently, unless where clearly stated otherwise. I have used only the sources, the data and the support that I have mentioned.

Munich, September 27, 2013

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Eliška Greplová