Entangled Atomic Ensembles - Generation and Applications

Diploma Thesis
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1 Introduction

Information is physical.
Rolf Landauer

Information is not independent of the physical laws that govern how it is stored, processed and transmitted [1]. The science of Quantum Information investigates the unique properties of quantum mechanics in this respect. Central results show that it is possible to take advantage of these properties in order to accomplish tasks which are intractable by systems based on classical physics. Apart from Quantum Information’s valuable contributions to elementary issues in quantum theory and from its impact on high sensitivity metrology, nowadays two major fields of research may be identified: quantum computation and quantum communication [2].

Quantum computers are information processing devices which exploit the full complexity of a many-particle wavefunction to solve computational problems, for which no efficient classical algorithms exist. Exemplary tasks are Shor’s algorithm for factorizing large numbers [3], Grover’s search algorithm [4] and the possibility to simulate quantum systems [5]. There are great practical difficulties in building large scale machines capable of implementing important computations. However, it has at least proved possible to build small demonstration devices in a vast range of candidate systems [6].

Quantum communication is the art of transferring a quantum state from one place to another. The most striking achievement is quantum cryptography [7], for which first applications are already commercially available. It allows two parties to establish a secret key, which is not just safe because of computational complexity like in classical proposals, but whose secrecy is guaranteed by the laws of quantum mechanics [8]. Other important quantum communication schemes are quantum teleportation [9] as the disembodied transfer of a quantum state or dense coding [10]. Together with quantum processing at separated nodes quantum networks can be envisaged [11].

Most of these protocols rely on entanglement between communication parties, a peculiarity at the heart of quantum mechanics’ non-locality which has gained a lot of attention lately. Quantum entanglement is a correlation between two or more quantum systems such that the description of their states has to be done with reference to each other even if they are spatially well separated. In 1935, this corner stone of quantum theory was discussed qualitatively for the first time by Einstein, Podolsky and Rosen [12] and later quantized by Bell in deriving his famous inequalities to distinguish quantum mechanics from classical local hidden variable models [13, 14]. Since then a lot of theoretical effort has been made in gaining a better understanding...
of entanglement \[15\], and from the 1960s on it has been demonstrated experimentally in several systems, for example between many photons \[16\], between a photon and an ion \[17\], between many ions \[18\] or between quantum dots \[19\].

For generation and distribution of entanglement, a well-designed and strong interaction between light as the natural carrier of quantum information and matter as the typical storage medium has to be obtained. The first promising setting was an atom put in a cavity: Cavity Quantum Electrodynamics (Cavity QED) \[20\]. This field of research has achieved spectacular success, but the formidable technical challenges of the strong coupling Cavity QED regime promoted search for alternative routes of atom-light interfaces. One of the most successful routes emerged with the recognition, that the use of collective quantum states of large atomic ensembles can also provide strong coupling to light.

The off-resonant scattering process of light at such a macroscopic object can remarkably be described as the interaction between a few properly defined bosonic modes, forward scattered light on the one and collective atomic excitations on the other side \[21\]. The Hamiltonian is easily designed to be any mixture of beamsplitter and parametric gain type, like for example the famous Faraday interaction. In the resonant limit a connection to the theory of electromagnetically induced transparency can be seen \[22\]. With Cesium vapor at room temperature as an atomic ensemble, for example a quantum memory for light \[23\] \[24\] and the first interspecies teleportation \[25\] \[26\] between light and matter have been demonstrated. The highly influential DLCZ-protocol \[27\] \[28\] for realizing quantum repeaters \[29\] is also based on atomic ensembles. Quantum repeaters create and store shorter-distance entanglement, and then connect the elementary entangled states to establish longer-distance entanglement via entanglement swapping and purification. They are necessary, since inevitable photon loss scales exponentially with the communication length, and it is therefore difficult to establish high quality entanglement directly over more than one hundred kilometers.

To deterministically generate entanglement between two atomic ensembles, one can now use a light beam that interacts with both of them \[30\]. This has experimentally been demonstrated with room-temperature Cesium vapor \[31\]. The main step is the measurement of the light field, which induces entanglement between the ensembles. Recent work shows a way, where the explicit measurement is not necessary any more, rather entanglement is generated by dissipation \[32\] \[33\]. At present, the system’s state can be kept entangled for 0.04 seconds this way.

The thesis at hand aims to contribute to this topic of deterministic entanglement generation. More precisely, the goal is to faster achieve more entanglement. To get there, we consider the benefit of supplementary passive operations, which can be implemented, for example, by means of time-dependent magnetic fields or waveplates during the interaction. This approach is inspired by similar ideas for other purposes \[34\] \[35\] \[36\], and we follow it in two steps. First, we investigate the question, whether in the optimal case these operations can be chosen independent of potential light measurement results. Otherwise, it would be a generalization of the linear feedback
schemes, which have proven successful in other systems [37, 38]. Second, we clarify the benefit of light measurements themselves and determine an actual optimal choice of supplementary operations on the ensembles.

The setups under consideration can be treated in the context of continuous variable quantum information [39, 40]; more specifically, the powerful Gaussian formalism is involved. In order to establish a firm basis for the following discussion, in Chapter 2 elements from the theory of Gaussian states and Gaussian operations are reviewed. With these tools at hand, following [25, 32, 41] in Chapter 3 a proper description of the atomic ensembles and the light field is introduced and the basic scattering interaction derived fully quantum mechanically. Chapter 4 then presents the basic idea of entangling two atomic ensembles by interaction with light. The latest advances are reviewed, and we motivate our strategies to achieve the goal of optimizing the protocol.

Chapter 5 is the first of the two main parts of the thesis. It answers the question of general feedback in our setup: To faster generate more entanglement within the Gaussian regime, the optimal choice of the additional operations does not depend on light measurement results. As the main result of the chapter, we show, that no form of feedback can help in general Gaussian conditional state's entanglement generation. Feedback may just be of use for conditioning, and its linear form is enough for this purpose.

After having ruled out their dependence on potential measurement results, in Chapter 6 we determine an optimal choice of additional passive Gaussian operations on the ensembles; this is the second main part of the thesis. We are guided to the statement, that static antipodal magnetic fields on the ensembles as applied in [25, 26, 31] are an optimal configuration. Also, we show that measurements of proper light quadratures further improve the scheme, although their results cannot be used for feedback.

In the end, Chapter 7 we study teleportation between atomic ensembles as an application of the optimal entangling procedure that was identified. On the basis of [30], two schemes are proposed and compared for a class of coherent input states. We find, that the better scheme is able to beat the quantum benchmark for state transfer [42, 43] for a small class of input states.
2 Fundamentals and Gaussian Formalism

An interesting approach to quantum information is based on continuous variables (CV) as opposed to the usual qubits or qudits [39, 40]. Herein, a very important role is played by the Gaussian formalism dealing with the broad class of Gaussian states and with Gaussian operations, that map the set of Gaussian states onto itself [44]. For the description of these states and operations methods of linear algebra on a finite dimensional phase space are mostly sufficient. Because of the therefore arising possibility to efficiently simulate Gaussian protocols of polynomially many operations on a classical computer, universal quantum computing is not possible with them. Moreover, entanglement purification cannot be achieved in a purely Gaussian setup [45], and so also error correction is not feasible [46]. On the other hand, unlike most general CV states, out of the same reasons the Gaussian ones are conveniently manageable in theory for example in describing light fields and atomic ensembles as for this thesis’ purpose, or even chains of ions in harmonic traps [47] and electrical circuits [48]. They can also be easily realized in laboratories and still efficiently and unconditionally enable many important tasks especially from quantum communication, like cryptography [49, 50], teleportation [51] or cloning [52]. A strength of the Gaussian settings is furthermore, that for optical systems nearly perfect detection schemes such as homodyne detection are experimentally readily available. In the long term, one aims to find clever ways to embed qubits into CV Hilbert spaces and unify this way the discrete and continuous variable approaches.

As the thesis at hand will make extensive use of the Gaussian formalism, the purpose of this chapter is to provide the reader with the necessary theoretical background. In the whole chapter basically just results are stated, for proofs or deeper explanations the interested reader is referred to the corresponding literature. Although they are probably well-known, the in the following most important basic concepts in quantum theory are reviewed in 2.1, such that in the process of introducing Gaussian states in 2.2 they then can be directly applied. In 2.3 Gaussian operations are characterized completely. After having presented the idea of entanglement measures in 2.4 with a special emphasis on negativity and EPR-variance, in Section 2.5 the main ideas of quantum state teleportation are reviewed in view of Chapter 7.
2.1 Basic Quantum Mechanics

As a starting point, the reader is supposed to be familiar with the formulation of quantum mechanics in state-vector language and the according Dirac bra-ket notation. The results stated here can be found in [1, 53] for Subsections 2.1.1 and 2.1.2, [1, 38] for 2.1.3 and [15] for 2.1.4.

2.1.1 The Density Operator

While in the usual state vector formulation of the Schrödinger picture a quantum system’s state is described by a vector $|\Psi(t)\rangle$ in a possibly infinite dimensional Hilbert space $H$, it can equivalently be characterized by the projection operator on the ket $|\Psi(t)\rangle$, $\rho(t) \equiv |\Psi(t)\rangle\langle\Psi(t)|$. This projection operator $\rho$ is called the density operator, and given a basis $\{e_i\}$ of $H$, it can be represented by a density matrix with entries $\langle e_i|\rho|e_j\rangle$.

All physical predictions that can be made with a state vector can be obtained from this operator as well. Its time evolution governed by the unitary $U(T)$ is given by

$$\rho(t + T) = U(T)\rho(t)U^\dagger(T),$$

and expectation values of observables $A$ are calculated via

$$\langle A\rangle(t) = \text{tr}\{\rho(t)A\}.$$

Subsystems $A$ and $B$ of a composite quantum system $AB$ with the joined system’s Hilbert space $H_{AB} = H_A \otimes H_B$ are conveniently described by the same formalism with the reduced density operators $\rho_A(t)$ and $\rho_B(t)$, which can be derived from the joint system’s density operator $\rho_{AB}(t)$ by applying the partial trace operator:

$$\rho_A(t) = \text{tr}_B\{\rho_{AB}(t)\}, \quad \rho_B(t) = \text{tr}_A\{\rho_{AB}(t)\}.$$

2.1.2 Pure and Mixed States

Suppose now, that the initial state of the system is not known exactly, but is with a probability $p_i$ in state $|\Psi_i\rangle$, where of course $\sum_i p_i = 1$. Such a statistical ensemble $\{p_i, |\Psi_i\rangle\}$ can easily be described by a generalization of the density operator introduced in the last Section:

$$\rho \equiv \sum_i p_i|\Psi_i\rangle\langle\Psi_i|. \quad (2.1)$$

This shows the power of the density operator formulation as well as the connection to statistical mechanics and with this said the origin of the operator’s name. Averaging over the different possible input states, this generalized density operator behaves formally in complete analogy to the equations given above. It can be shown, that an
operator $\rho$ on $\mathcal{H}$ is the density operator associated with some ensemble $\{p_i, |\Psi_i\rangle\}$, if and only if

i) $\text{tr}\{\rho\} = 1$ (and is, for this to make sense, bounded) because of conservation of probability,

ii) $\rho^\dagger = \rho$, i. e. it is Hermitian/self-adjoint and

iii) $\rho \geq 0$, i. e. it is positive semidefinite (i. e. that its spectrum is non-negative, or, equivalently, $\langle \phi | \rho | \phi \rangle \geq 0 \ \forall | \phi \rangle \in \mathcal{H}$).

One says, that the underlying physical system is in a \textit{pure state}, if all but one of the $p_i$ vanish like in the previous section. Otherwise, the system is said to be in a \textit{mixed state}, which differs from a superpositioned pure state in not showing interference between the states of the mixture. A measure for the degree of mixture is the \textit{purity} $\mu \equiv \text{tr}\{\rho^2\}$ with $0 \leq \mu \leq 1$. $\mu = 1$ just for pure states, as only they are projectors and therefore $\rho^2 = \rho$ then.

### 2.1.3 Measurements

The state, represented by the density operator, describes an observer’s knowledge of a quantum system and is therefore changed in a measurement process. So, an \textit{a-priori state} $\rho(t)$ before a measurement of duration $T$ is taken to a \textit{conditional} or \textit{a-posteriori state} $\rho_r(t + T)$ afterwards, which is conditioned on the probabilistic result $r$. Also, by averaging over all possible results, an \textit{unconditional state} $\rho(t + T)$ can be defined; it accounts for the case of ignoring the result $r$ or having the environment measure in open systems.

The most general measurement which can be applied to a quantum system has to be described in terms of operations, but in this thesis the special case of a \textit{POVM measurement} (positive operator-valued measure) will suffice. A POVM is any set of positive operators $E_r$, which is a resolution of the Hilbert space’s identity and therefore fulfills the completeness condition $\sum_r E_r = 1_{\mathcal{H}}$. This makes sure, that the results’ probabilities sum to 1. By writing these so-called \textit{effects} $E_r$ as $M_r^\dagger M_r$, one gets the set of \textit{measurement operators} $\{M_r\}$. They then allow for a description of the conditional

$$\rho_r(t + T) = \frac{M_r\rho(t)M_r^\dagger}{p_r},$$

and unconditional state

$$\rho(t + T) = \sum_r p_r\rho_r(t + T) = \sum_r M_r\rho(t)M_r^\dagger$$  \hspace{1cm} (2.3)
after having obtained a result \( r \) with probability \( p_r = \text{tr}\{E_r\rho(t)\} \).

The well-known \textit{projective measurement}, which corresponds to the idea of measuring an observable \( N \), is then included as a special case. This is due to the spectral theorem for normal \( ([N,N^\dagger] = 0) \) operators, that states

\[
N = \sum_r r P_r
\]

for projectors

\[
P_r = \sum_s |r,s\rangle\langle r,s|
\]

(with \( s \) for degeneracy) being orthonormal \( P_r P_r' = \delta_{rr'} \), and fulfilling \( \sum_r P_r = 1_H \). The choice \( M_r = P_r \) and \( E_r = P_r^\dagger P_r = P_r \) then shows, how the projective measurements fit into the POVM formalism.

By projectively measuring on the environment \( E \) of an open system \( S \) with \( M_{Er} = P_{Er} = (|r\rangle\langle r|)_E \), where \( \{|r\rangle\} \) is an orthonormal basis of \( E \), and using the update formula for the unconditional state, \( (2.3) \), it can be seen, that for the system’s state measuring on the environment and ignoring the result is equivalent to tracing out the environment:

\[
\rho_{SE}(t+T) = \sum_r P_{Er}\rho_{SE}(t)P_{Er}
= \sum_r (\langle r|\rho_{SE}(t)|r\rangle)_{S} \otimes (|r\rangle\langle r|)_E.
\] (2.4)

\subsection*{2.1.4 Separability and Entanglement}

Whereas a theory of general multipartite entanglement is currently supposed to be very demanding, for this thesis it is enough to deal with the much easier and far better understood case of entanglement of a bipartite quantum system. Here, a pure joint qubit- or, more general, qudit state \( |\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) is called \textit{separable}, if it can be written as a product of subsystem wave functions \( |\Phi_A\rangle \in \mathcal{H}_A \) and \( |\Phi_B\rangle \in \mathcal{H}_B \):

\[
|\Psi_{AB}\rangle = |\Phi_A\rangle \otimes |\Phi_B\rangle.
\] (2.5)

If a state is not separable, it is called \textit{entangled}, with different degrees distinguished by so-called \textit{entanglement measures}, see \( \underline{2.4} \). The physical interpretation of entangled pure states is therefore, that the joint state is completely known, while one doesn’t have all or, in the maximally entangled case, any information about the states the subsystems are in (see Section \( \underline{2.2.4} \) for examples).

For mixed states, the idea of separability is, that one should be able to express or approximate the composite state as a mixture of product states. Out of mathematical subtleties for infinite dimensional systems like the Gaussian ones considered mainly in this thesis, the rigorous definition is a little more complicated: A mixed state \( \rho_{AB} \) defined on a Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \) is said to be separable, if it is a limit of a sequence of finite convex combinations of product states,

\[
\| \rho_{AB} - \sum_i p_i^n \rho_{A_i}^{n_i} \otimes \rho_{B_i}^{n_i} \| \to 0 \quad (n \to \infty),
\] (2.6)
where $\rho_{Ai}^n$ and $\rho_{Bi}^n$ are, if $\dim \mathcal{H}_{AB} < \infty$, pure and otherwise mixed physical states on the subsystems $A$ and $B$, respectively, $p_i^A \geq 0$ and $\sum_i p_i^A = 1$. Furthermore, $\|A\| \equiv \text{tr}\{\sqrt{A^\dagger A}\}$ denotes the trace norm, which is equal to the sum of the absolute values of the eigenvalues of $A$, if $A$ is Hermitian.

To decide in general, if a state is separable or not, is very difficult and was shown to be formally NP-hard [54]. However, besides the historically important idea to use Bell inequalities [14] for this task, Peres suggested a stronger necessary criterion for separability of a generally mixed bipartite quantum state based on the positivity of its partial transpose, the important PPT-criterion [55]. By defining for a state $\rho_{AB}$ the partial transpose $\rho_{AB}^{T_A}$ with respect to a subsystem, here without loss of generality to $A$, as the operator with matrix elements $\langle ij | \rho_{AB}^{T_A} | kl \rangle = \langle kj | \rho_{AB} | il \rangle$ in some product basis, it can be seen, that a separable state’s

$$\rho_{\text{sep}} = \sum p_i \rho_{Ai} \otimes \rho_{Bi}$$

partial transpose is

$$\rho_{\text{sep}}^{T_A} = \sum p_i (\rho_{Ai})^T \otimes \rho_{Bi}.\$$

Here, $T$ denotes transposition and mathematical subtleties mentioned above are disregarded to just give the main idea. $\rho_{\text{sep}}^{T_A}$ has to be positive semidefinite, because transposition preserves the spectrum (and for this statement the choice of basis above doesn’t matter) and $\rho_{Ai}$ is positive semidefinite as a physical state. Therefore, if a state is separable, its partial transpose necessarily has to be positive semidefinite, and, in turn, if a state’s partial transpose is not positive semidefinite, this is a sufficient condition for entanglement.

Nevertheless, the PPT is in general not sufficient for separability due to the existence of so-called PPT bound entangled states.

### 2.2 Gaussian States

The presentation of this Section is along the lines of [44].

#### 2.2.1 Symplectic Tools

Systems of bosonic modes are considered, for which canonical operators are defined in terms of annihilation $a$ and creation $a^\dagger$ operators as

$$x = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad p = \frac{1}{\sqrt{2}i} (a - a^\dagger).$$

(2.7)

The phase space of $n$ such modes is a $2n$-dimensional real symplectic vector space (in this thesis of finite dimension) equipped with the symplectic form $\sigma$, i.e., up to an isomorphism it is the vector space $\mathbb{R}^{2n}$ and $\sigma$ a non-degenerate ($\sigma(\xi, \eta) = 0$ for all
\[ \eta \in \mathbb{R}^{2n} \Rightarrow \xi = 0 \) antisymmetric \((\sigma^T = -\sigma)\) bilinear form on it. After having chosen a suitable basis, this symplectic form is given by a matrix (which is in this thesis not notationally distinguished from the form) with elements \(\sigma_{kl}\),

\[
\sigma = \begin{pmatrix}
\sigma_2 & \sigma_2 & \cdots & \sigma_2 \\
-\sigma_2 & \sigma_2 & \cdots & \sigma_2 \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_2 & -\sigma_2 & \cdots & \sigma_2
\end{pmatrix},
\]

(2.8)

where \(\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\), governing the canonical commutation relations (CCR) for the according Hilbert space \(\mathcal{H}\)'s 2n canonical operators \(R = (x_1, p_1, \ldots, x_n, p_n)^T\):

\[
[R_k, R_l] = i\sigma_{kl}1
\]

(2.9)

with \([A, B] \equiv AB - BA\) denoting the commutator.

The set of linear maps \(S\) on phase space, for which \(S\sigma S^T = \sigma\), form the symplectic group \(Sp(2n, \mathbb{R})\) of symplectic or canonical transformations, and together with translations the affine symplectic group. For \(S \in Sp(2n, \mathbb{R})\), it holds that also \(S^T \in Sp(2n, (\mathbb{R})\) and \(S^{-1} = \sigma S^T \sigma^{-1}\).

Finally, a theorem by Williamson \[56\] is needed in the following: Every real and positive definite \(2n \times 2n\) matrix \(C\) can be diagonalized such that

\[
SCS^T = \text{diag}(s_1, s_1, \ldots, s_n, s_n),
\]

(2.10)

where \(S \in Sp(2n, \mathbb{R})\) and \(s_i > 0\). The symplectic eigenvalues \(\{s_i\}\) appear also as (regular) eigenvalues of the matrix \(i\sigma^{-1}C\), such that \(\text{spec}\{i\sigma^{-1}C\} = \{\pm s_i\}\).

### 2.2.2 Phase Space Quantization

In Weyl- or phase space quantization \[57\], a quantum system is described by complex-valued functions on the phase space, \(g(\xi)\), associated with the ”usual” operators on \(\mathcal{H}\), \(g\), by a change of representations. For trace-class operators (operators \(g\) with \(\text{tr}\{\sqrt{g^*g}\} < \infty\)), this one-to-one correspondence is achieved through

\[
g(\xi) = \text{tr}\{W_\xi g\},
\]

and the inverse through the Fourier-Weyl-transform

\[
g = (2\pi)^{-n} \int g(\xi)W_{-\xi}d^{2n}\xi.
\]

\(W_\xi\) denotes a Weyl or translation operator as part of a Weyl system, i. e. a family of unitaries \(W_\xi\) satisfying the Weyl relations \(W_\xi W_\eta = e^{-\frac{i}{2} \xi^T \sigma \eta} W_{\xi + \eta}\). It corresponds,
up to an irrelevant phase, to a translation in phase space, because in the Schrödinger representation

\[ (W(\xi_q, \xi_p)\Psi)(x) = e^{-i\frac{\xi_q x + \xi_p x}{2}}\Psi(x + \xi_q). \]

The function associated with the density matrix is called the characteristic function of the state,

\[ \chi(\xi) = \text{tr}\{W_\xi \rho\}. \quad (2.11) \]

This definition corresponds to symmetric ordering of the quantum mechanical operators, and its (symplectic) Fourier transform is called the Wigner function:

\[ W(\xi) = (2\pi)^{-2n} \int e^{i\xi^T \sigma \eta} \chi(\eta) d^{2n} \eta. \quad (2.12) \]

It is normalized, and although its marginal distributions with respect to \( q \) and \( p \) return their probability distributions respectively, it can take on negative values. The same construction for normal or anti-normal operator ordering yields the similar P- and Q-function.

### 2.2.3 General Definition

The mean values and the generalized second moments of the canonical variables

\[ d_k \equiv \langle R_k \rangle \equiv \text{tr}\{\rho R_k\} \]
\[ \gamma_{kl} \equiv \text{tr}\{\rho\{R_k - d_k \mathbb{1}, R_l - d_l \mathbb{1}\}\} \]
\[ = \text{tr}\{\rho(R_k - d_k \mathbb{1})(R_l - d_l \mathbb{1})\} - i\sigma_{kl}, \quad (2.13) \]

where \( \{ \ldots \} \) denotes the anti-commutator and \( \langle A \rangle \equiv \text{tr}\{\rho A\} \) the expectation value of an operator \( A \), form the displacement vector \( d \) and the symmetric covariance matrix \( \gamma \). Like in probability theory, they can be calculated by the derivatives of the characteristic function.

The diagonal elements of the covariance matrix yield twice the variances,

\[ \gamma_{kk} = 2\text{tr}\{\rho(R_k - \langle R_k \rangle)^2\} = 2\text{tr}\{\rho(R_k^2 - \langle R_k \rangle^2)\} = 2(\Delta R_k)^2, \]

and its off-diagonal elements the correlation coefficients \(-1 \leq C_{R_k R_l} \leq 1\) of the canonical operators known in statistics,

\[ C_{R_k R_l} = \frac{\gamma_{kl}}{\sqrt{\gamma_{kk} \gamma_{ll}}}. \]

Pure separable states don’t have any correlations.

A **Gaussian state** is defined as a quantum state having a Gaussian Wigner function or, equivalently, a Gaussian characteristic function:
\[ W(\xi) = ce^{-(\xi-d)^T\gamma^{-1}(\xi-d)}, \quad c \equiv \pi^{-n}\det(\gamma)^{-1/2}, \tag{2.14} \]

and is therefore completely determined by its displacement vector and its covariance matrix.

For Gaussian states, the condition \( \rho \geq 0 \) for the density operator to describe a physical ensemble (see Section 2.1.2) is equivalent to

\[ \gamma + i\sigma \geq 0, \tag{2.15} \]

by complex conjugation as well to \( \gamma - i\sigma \geq 0 \) and to the condition, that the *symplectic eigenvalues* of the covariance matrix have to be larger than or equal to one. This can be seen as a basis-independent formulation of Heisenberg’s uncertainty relation for many canonical degrees of freedom; e. g. in the case of a diagonal single-mode covariance matrix one recovers the well-known form

\[ (\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}. \tag{2.16} \]

Similarly, the concept of purity can be related to the Gaussian case \[ 58],

\[ \mu = 1/\sqrt{\det(\gamma)}, \]

and a state is pure iff \( \det(\gamma) = 1 \).

An investigation of entanglement properties shows, that a bipartite Gaussian state with covariance matrix \( \gamma_{AB} \) is separable iff there are subsystem covariance matrices \( \gamma_A \) and \( \gamma_B \) such that

\[ \gamma_{AB} \geq \gamma_A \oplus \gamma_B. \]

Because displacements of Gaussian states can be removed by just local unitary operations and entanglement is the non-local property per se, the entanglement existent in a Gaussian state is completely determined by the covariance matrix.

### 2.2.4 Example: Two Mode Squeezed States

A single-mode minimum uncertainty state is a state saturating (2.16) exactly. If, in addition \( \langle (\Delta x)^2 \rangle = \langle (\Delta p)^2 \rangle = 1/2 \), it is called a coherent state \( \ket{\alpha} \). It is an eigenstate of the annihilation operator, \( a\ket{\alpha} = \alpha\ket{\alpha} \), and it holds, that

\[ d_{coh} = \sqrt{2} \begin{pmatrix} \Re(\alpha) \\ \Im(\alpha) \end{pmatrix}, \quad \gamma_{coh} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.17} \]
When the displacement vector vanishes, $d = 0$, a state is called \textit{centered}, and the centered coherent state is referred to as the \textit{vacuum $|0\rangle$}, see Figure 2.1a. A single-mode \textit{squeezed state} is a state satisfying $(\Delta y)^2 < 1/2$ in one quadrature $y = x/p$. The other quadrature then is said to be antisqueezed \cite{59,60}. In this spirit, a multi-mode Gaussian state with covariance matrix $\gamma$ is called coherent, if $\gamma$ just has eigenvalues $1$, and squeezed, if it has at least one eigenvalue smaller than $1$.

Since it can be shown, that every entangled Gaussian state is squeezed, these states play an important role. An example is the pure minimum uncertainty \textit{two mode squeezed state} $|\Phi_{\text{tms}}(r)\rangle$ with \textit{squeezing parameter} $r \in \mathbb{R}$ given by the covariance matrix

$$
\gamma_{\text{tms}} = \begin{pmatrix}
\cosh(r) & 0 & \sinh(r) & 0 \\
0 & \cosh(r) & 0 & -\sinh(r) \\
\sinh(r) & 0 & \cosh(r) & 0 \\
0 & -\sinh(r) & 0 & \cosh(r)
\end{pmatrix} \equiv \begin{pmatrix} A_r & C_r \\
C_r & A_r \end{pmatrix}.
$$

(2.18)

Changing the basis to the EPR eigenbasis $R_{\pm} = (x_+, p_+, x_-, p_-)^T$ with

$$
\begin{pmatrix} x_+ \\ p_+ \\ x_- \\ p_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_A \\ p_A \\ x_B \\ p_B \end{pmatrix},
$$

(2.19)

the covariance matrix is, see Figure 2.1b

$$
\gamma_{\text{tms},\pm} = \begin{pmatrix}
e^r & 0 & 0 & 0 \\
0 & e^{-r} & 0 & 0 \\
0 & 0 & e^{-r} & 0 \\
0 & 0 & 0 & e^r
\end{pmatrix}.
$$

(2.20)
In this form, it is easily seen, that $x_-$ and $p_+$ are squeezed, while $x_+$ and $p_-$ are antisqueezed. E.g. the correlation $C_{x_1 x_2} = \frac{\gamma_{13}}{\sqrt{\gamma_{11} \gamma_{33}}} = \frac{\sinh(r)}{\cosh(r)} \to 1 \ (r \to \infty)$ shows the non-local property of this entangled state. In the Fock basis, it can be written as

$$|\Phi_{\text{tms}}(r)\rangle = \frac{1}{\cosh(r)} \sum_{k=0}^{\infty} \tanh^k(r) |k\rangle \otimes |k\rangle,$$

and for $r \to \infty$, this is the (unnormalized) optimally entangled EPR state $|\Phi\rangle \propto \sum_{k=0}^{\infty} |k\rangle \otimes |k\rangle$ [12]. It is the CV analogon of $|\beta_1\rangle$ of the optimally entangled Bell states for qubits

$$|\beta_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \quad |\beta_2\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle),$$

$$|\beta_3\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle), \quad |\beta_4\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle).$$

### 2.3 Gaussian Operations

This section starts with a characterization of unitary Gaussian operations [44], then extends the concept to trace-preserving completely positive Gaussian maps [1, 44, 61] and in the end to the most general form of not necessarily trace-preserving, but still completely positive Gaussian maps [45].

#### 2.3.1 Unitary Gaussian Operations

A unitary operation $E$ acting on density operators $\rho$ is a map of the form

$$E: \rho \mapsto U \rho U^\dagger,$$

where $U$ is a unitary operator on $\mathcal{H}$. Accordingly, a unitary Gaussian operation $G$ is one of this form mapping Gaussian states to Gaussian states. On the level of Weyl operators $W_\xi$, these Gaussian ones correspond to transformations of the form

$$W_\xi \mapsto e^{y(\xi)} W_{S^{-1}\xi},$$

where $S \in Sp(2n, \mathbb{R})$ and $y(\xi) = i\xi D$ with $D \in \mathbb{R}^{2n}$ is a linear form. They, in turn, correspond to affine symplectic transformations on phase space. Leaving out the for this thesis' point unimportant, as always locally applicable, displacements $D$, the unitary Gaussian operations $G$ may then be written as in the Heisenberg picture acting on the canonical operators $R$ like $R \mapsto R' \equiv SR$.

The other way round, it can be shown, that for every operation of the latter form to be physical, i.e. to preserve the CCR for $R'$, $S$ has to be symplectic, and that
for every symplectic S there is a unitary operation and therefore a unitary $U_{S^{-1}}$ such that $W_\xi \mapsto U_{S^{-1}}W_\xi U_{S^{-1}}^{-1} = W_{S^{-1}\xi}$. So, for a Gaussian input state with characteristic function $\chi_{in}(\xi)$ (2.14), this operation with $S$ changes the function (2.11) to

$$\chi_{out}(\xi) = \text{tr}\{\rho W_{S^{-1}\xi}\} = \chi_{in}(S^{-1}\xi)$$

$$= e^{i(S^{-1}\xi)^T \sigma d - \frac{1}{4}(S^{-1}\xi)^T \sigma \sigma^T S^{-1}\xi}$$

$$= e^{i\xi^T \sigma S d - \frac{1}{4}\xi^T \sigma S \gamma S^T \sigma^T \xi},$$

where $(S^{-1})^T \sigma = \sigma \sigma^T \sigma = \sigma S$, because $\sigma^T = \sigma^{-1}$, was used, and so the post-operation output state is also Gaussian with

$$\gamma \mapsto \gamma' \equiv S \gamma S^T$$

$$d \mapsto d' \equiv S d.$$  (2.23)

Furthermore, on group generator level it can be seen, that every quadratic Hermitian Hamiltonian $H = R^T M R$, $M \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, gives rise to a symplectic transformation. All together, one can state, that up to mere translations the unitary Gaussian operations can be written as multiplication with a symplectic matrix and contain evolutions of quadratic Hamiltonians.

Given the Hamiltonian, the according symplectic $S$ can be calculated as follows: The time evolution of the canonical operators is given by Heisenberg’s equation as

$$\partial_t R(t) = i[H, R(t)] = i[R(t)^T M R(t), R(t)].$$

Considering this equation componentwise and using the CCR, one obtains

$$\partial_t R_j = i \sum_{kl} m_{kl} R_k R_l, R_j = i \sum_{kl} m_{kl} ([R_k, R_j] R_l + R_k [R_l, R_j])$$

$$= - \sum_{kl} m_{kl} (\sigma_{kj} R_l + R_k \sigma_{lj}) = - \sum_{kl} (-\sigma_{jk} m_{kl} R_l + m_{kl} \sigma_{lj} R_k)$$

$$= \sum_l (\sigma M)_{jl} R_l - \sum_k (M \sigma)_{kj} R_k = \sum_l ((\sigma M)_{jl} - (\sigma^T M^T)_{jl}) R_l.$$  

Therefore,

$$\partial_t R = (\sigma M + \sigma M^T) R = GR,$$  (2.24)

where $G \equiv \sigma M + \sigma M^T$, which is solved by setting $S \equiv e^{Gt}$:

$$R(t) = e^{Gt} R(0) = SR(0).$$  (2.25)
Moreover, every symplectic transformation $S$ can by the polar decomposition be written as $S = PA$ with $P \in K(n) \equiv Sp(2n,\mathbb{R}) \cap SO(2n)$ and $A \in \Pi(n) \equiv \{ S \in Sp(2n,\mathbb{R}) | S = S^T, S > 0 \}$. Elements of these subgroups of $Sp(2n,\mathbb{R})$ are called passive and active transformations, respectively. All elements in these sets correspond to generating Hamiltonians, which is not true for an arbitrary element of $Sp(2n,\mathbb{R})$. The passive operations are exactly those, whose Hamiltonians commute with number operators and therefore preserve particle numbers, such as beam splitters or mirrors in the optical setting; an example of active operations are squeezers.

### 2.3.2 Gaussian Channels

In general, a quantum channel $\mathcal{E}$ is a trace-preserving and completely positive (CP) operation, that maps density operators $\rho_A$ of a quantum system $A$ to density operators $\rho_B$ of system $B$,

$$\rho_A \mapsto \mathcal{E}(\rho_A) = \rho_B. \tag{2.26}$$

A map $\mathcal{E}$ is called trace-preserving, if $\text{tr}\{\mathcal{E}(\rho_A)\} = \text{tr}\{\rho_A\} = 1$, and CP, if not only $\forall \rho_A \geq 0 : \mathcal{E}(\rho_A) \geq 0$ (which is called positive), but also for an additional system $C$ of arbitrary dimension $\forall \rho_{AC} \geq 0 : (1_C \otimes \mathcal{E})(\rho_{AC}) \geq 0$. An example of a map which is not CP although positive is the transposition introduced in 2.1.4. There, the PPT-criterion relies exactly on this feature.

From now on, just channels from a system to itself are considered; they can be used to describe for example a system’s time-evolution. An alternative description can in this case be given in terms of a physical system $S$ embedded in an environment $E$. The channel action is then given by the reduction to the state of $S$ after having had a unitary $U$ acting on the joint system:

$$\rho_S \mapsto \mathcal{E}(\rho_S) = \text{tr}_E\{U(\rho_S \otimes \rho_E)U^\dagger\}. \tag{2.27}$$

Yet another description of quantum channels is formulated in terms of Kraus-operators \[62\], but already from the definition here it becomes clear, that quantum channels allow for open systems, coupling to the environment and therefore noise to enter the system. In this sense they are closely related to the theory of master equations \[60\]. They generalize the unitary operations from the previous section.

Just considering the quantum channels that map Gaussian states to Gaussian states leads to the Gaussian channels $\mathcal{G}$. This set can be obtained by relaxing (2.3.1) to

$$W_\xi \mapsto e^{y(\xi)}W_\xi \tag{2.28}$$

with $y(\xi) = -1/2\xi^TY\xi$ now a quadratic form, $X, Y \in \mathbb{R}^{2n \times 2n}$ and $Y$ symmetric. Here, mere translations are, once again, skipped, and the additional constraint

$$Y + i\sigma - iX^T\sigma X \geq 0 \tag{2.28}$$
is imposed. Compared to the unitary Gaussian operations, if $X$ is not symplectic, at least a minimal $Y$ has to be added to receive a valid transformation.

Apart from the Schrödinger picture’s channel characterization used in the beginning of the chapter, like in (2.23) the matrices $X$ and $Y$ allow here for a convenient description in terms of covariance matrices linked to the general case’s interpretation (2.27). Let $\gamma_S$ and $\gamma_E$ be the covariance matrices corresponding to Gaussian states of $n$ system and $m$ environment modes, respectively, and $S \in Sp(2(n + m), \mathbb{R})$, then there exist $X$ and $Y$ as specified above, such that

$$\gamma_S \mapsto [S(\gamma_S \otimes \gamma_E)S^T]_{2n} = X^T \gamma_S X + Y,$$

where $[...]_{2n}$ denotes the restriction to the upper submatrix of size $2n \times 2n$. This corresponds to taking the partial trace on density operator level. It preserves the Gaussian property of the state, whereas the symplectic matrix here expresses the unitary’s involvement from above.

$X$ can be seen as to serve for rotation or amplification, and the $Y$ contribution is a noise term. In addition to the necessary quantum noise $Y$ according to (2.28), there might be added some classical noise $Y' \geq 0$. For example, in a classical noise channel $X = 1$ and $Y \geq 0$, which corresponds to a random displacement according to a Gaussian distribution in the Schrödinger picture

$$\mathcal{G}(\rho_S) = \frac{1}{4\pi \sqrt{\det(Y)}} \int W_\xi \rho_S W_\xi^\dagger e^{-\frac{1}{4} \xi^T Y^{-1} \xi} d^2 \xi,$$

see [61] and references therein for this and other examples like the photonic loss channel.

### 2.3.3 General Gaussian Operations

The most general class of all allowed physical actions can be obtained by dropping the condition of trace preservation in the former section, so that they correspond to the set of all CP maps. The main goal of this ultimate generalization is the inclusion of measurements in the class of quantum operations.

A good way to characterize them is to use an isomorphism between these CP maps $\mathcal{E}$ on density operators of a system with Hilbert space $\mathcal{H}$ and positive operators $E$, i. e., according to 2.1.1, unnormalized physical states, on $\mathcal{H} \otimes \mathcal{H}$ [63]. This is established via defining

$$E_{12} \equiv (\mathcal{E} \otimes 1)(|\Phi\rangle_{12}\langle \Phi|),$$

where $|\Phi\rangle$ is the optimally entangled state introduced in 2.2.4 and number indices in this section label the part of the tensor product the operators are defined on.

Restricting the consideration, once again, just to general Gaussian operations $\mathcal{G}$ that map Gaussian states $\rho_{\gamma,d}$ to Gaussian states $\rho'_{\gamma',d'}$, 

\[ G : \rho_{\gamma,d} \mapsto \rho_{\gamma',d'}, \] (2.31)

and writing the due to the isomorphism according Gaussian operator/state \( G \) with covariance matrix \( \Gamma \) and displacement vector \( D \),

\[
\Gamma = \begin{pmatrix} \Gamma_1^T & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},
\]

the important complete characterization of \( G \) on the level of displacement vectors and covariance matrices

\[
\gamma' = \tilde{\Gamma}_1 - \tilde{\Gamma}_{12} \frac{1}{\Gamma_2 + \gamma} \tilde{\Gamma}_{12}^T \\
d' = D_1 + \tilde{\Gamma}_{12} \frac{1}{\Gamma_2 + \gamma} (D_2 + d)
\]

(2.32)

can be given. Here, \( \tilde{\Gamma} \equiv (I \oplus \Lambda) \Gamma (I \oplus \Lambda) \) with \( \Lambda \equiv \text{diag}(1, -1, 1, -1, \ldots) \), i.e. the diagonal matrix with entries \( 1, -1, \ldots \) on the main diagonal, and the partitioning in these definitions always corresponds to the natural given by \( H \otimes H \).

For example, directly from (2.30), it can be seen, that in general to \( E = I \), there corresponds the optimally entangled state \( E = |\Phi\rangle\langle\Phi| \) with \( \Gamma = \lim_{r \to \infty} \gamma_{\text{tms}}(r) \) and \( D = 0 \). The Gaussian channels are included in this formalism by the choice

\[
\Gamma = \lim_{r \to \infty} \begin{pmatrix} X^T A_r X + Y & X^T C_r \\ C_r X & A_r \end{pmatrix},
\]

with \( A_r, C_r \) and \( \gamma_{\text{tms}} \) from (2.2.4).

New operations in this broad class are the non-trace-preserving Gaussian measurements \( G_M \). Especially those of subsystem \( B \) of a bipartite system \( AB \) with POVMs of the form \( \{ |\gamma_M, d_M \rangle_B \langle \gamma_M, d_M | : d_M \in \mathbb{R}^{2n} \} \) are needed. If one is just interested in the other subsystem's state, \( B \) can be traced out after the measurement. With the formalism derived above, in this sense subsystem \( A \)'s state after the measurement with covariance \( \gamma_A' \) and displacement \( d_A' \) can then be given in terms of the joint input state with

\[
\gamma_{AB} = \begin{pmatrix} \gamma_A^T \\ \gamma_C^T \end{pmatrix} \quad \text{and} \quad d_{AB} = \begin{pmatrix} d_A \\ d_B \end{pmatrix}:
\]

\[
d_A' = d_A - \gamma_C (\gamma_B + \gamma_M)^{MP} (d_B - d_M) \\
\gamma_A' = \gamma_A - \gamma_C (\gamma_B + \gamma_M)^{MP} \gamma_C.
\]

(2.33)

Here, \( (...)^{MP} \) denotes the Moore-Penrose-pseudoinverse \[64\] coinciding with the usual matrix inverse, if the latter exists. The in this thesis very important homodyne measurements of, without loss of generality, the \( x \)-quadrature of a single-mode system \( B \) correspond formally to the POVM with \( \gamma_M = \lim_{g \to \infty} \text{diag}(1/g, g) \). For them,
can either be calculated with the usual inverse for finite $g$ taking the limit in the very end after all other calculation steps, or by realizing that in this case $(\gamma_B + \gamma_M)^{MP} = (\pi \gamma_B)^{MP} = \begin{pmatrix} 1/b & 0 \\ 0 & 0 \end{pmatrix}$ for $\pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\gamma_B = \begin{pmatrix} b & * \\ * & * \end{pmatrix}$ [65]. Note, that if the states are separable, $\gamma_C = 0$ and therefore measurement of subsystem $B$ doesn’t influence the state of subsystem $A$.

2.4 Entanglement Measures

In this section, the idea of measuring different degrees of entanglement of bipartite quantum states is picked up again and applied to Gaussian states. More details about the general first section can be found in [44, 66, 67], the negativity is introduced in [68], and the EPR-variance in [69, 70].

2.4.1 General Properties

Entanglement measures $E$ are functionals from the set of all states $\{\rho\}$ to the positive real numbers. It has been widely agreed on, that for such a functional to be a measure in general the *monotonicity* property has to be fulfilled, meaning that LOCC operations don’t increase it. More precisely, for a LOCC operation producing an ensemble $\{p_i, \rho_{i \text{out}}\}$ out of an initial state $\rho_{i \text{in}}$ \[ \sum_i p_i E(\rho_{i \text{out}}) \leq E(\rho_{i \text{in}}). \] (2.34)

Several other properties like *convexity* $(E(\sum_i p_i \rho_i) \leq \sum_i p_i E(\rho_i))$, *faithfulness* $(E(\rho) = 0$ iff $\rho$ is separable), *additivity* $(E(\rho_A \otimes \rho_B) = E(\rho_A) + E(\rho_B))$ or *asymptotic continuity* (meaningful properties for systems of infinitely many modes) are desirable, but not all fulfilled by all measures.

For pure states $\rho_{AB}$, for example, the requirements monotonicity, convexity and asymptotic continuity uniquely define the *entropy of entanglement* $E_e(\rho_{AB})$. It is the *von Neumann-entropy* $S$ of one of the reduced density operators, $E_e(\rho_{AB}) = S(\rho_A) = -\text{tr}\{\rho_A \log_2 \rho_A\}$ and gives basically all information about the possibility to transfer a number of copies of $\rho_{AB}$ to other pure states by means of LOCC in an asymptotic sense. Therefore, it is *operationally meaningful*.

For mixed states, there are also entanglement measures closely related to transformation properties. The *entanglement of formation*, or its renormalized version, the *entanglement cost*, and the *distillable entanglement* quantify, respectively, the asymptotic pure state entanglement needed to create $\rho_{AB}$ and that which can be distilled from it, both using LOCC. These measures don’t coincide for mixed states, and till today no easy way is known to calculate them. But because numerical ease is necessary for this thesis, instead the *(logarithmic) negativity* and the *EPR-variance* are mostly
used as measures, although they don’t have an immediate operational meaning like the above.

2.4.2 (Logarithmic) Negativity

As the PPT criterion states (see 2.1.4) for a bipartite state $\rho_{AB}$ of $n+m$ modes, that the partial transpose of separable states has just positive eigenvalues, and as the trace norm sums all eigenvalues, one can define the negativity

$$\mathcal{N} = \|\rho^{T_A}\| - \frac{1}{2}$$

and show, that by summing the negative eigenvalues it measures how close the state is to a positive definite one. It turns out to fulfill the criteria for an entanglement measure given in the former section.

Closely related is the logarithmic negativity

$$E_N = \log\|\rho^{T_A}\|. $$

In the Gaussian case, with the information of (2.15), the PPT criterion may be reformulated: Let $\gamma_{AB}^{T_A}$ be the covariance matrix corresponding to $\rho_{AB}^{T_A}$, then if $\rho_{AB}$ is separable, necessarily $\gamma_{AB}^{T_A}$ fulfills the uncertainty relation $\gamma_{AB}^{T_A} \geq i\sigma$. That means, that negative eigenvalues of $\rho_{AB}^{T_A}$ correspond to negative eigenvalues of $\gamma_{AB}^{T_A} - i\sigma$. These are those of the symplectic eigenvalues of the still positive definite $\gamma_{AB}^{T_A}$, $\{s_i\}$, that are smaller than 1, see (2.10) and thereafter.

By noting, that partial transposition corresponds to time reversal and therefore to a sign-flip for the momenta of the transposed subsystem,

$$\gamma_{AB}^{T_A} = (Z_n \oplus I_m)\gamma_{AB}(Z_n \oplus I_m)$$

with $Z \equiv \text{diag}(1, -1, 1, -1, ...)$ and $\gamma_{AB}$ the original covariance matrix. With this, in the Gaussian case the logarithmic negativity can then be calculated as

$$E_N = -\sum_{i=1}^{n+m} \log(\min\{s_i, 1\}).$$

Because for Gaussian states of $1 \times n$ modes no PPT bound entangled states exist [44], the PPT criterion turns out to be both necessary and sufficient in this case. Therefore, in the for this thesis important Gaussian states of $1 \times 1$ modes the (logarithmic) negativity is faithful. On the other hand, it turned out, that the negativity is not additive, whereas the logarithmic negativity is not convex (see also [71]).

2.4.3 EPR-Variance

Another entanglement measure used throughout this thesis is the EPR-variance $\Delta_{EPR}(\rho_{AB})$ of a bipartite CV state $\rho_{AB}$ of $1 \times 1$ modes. For generalized EPR-operators
\[ u \equiv |a|x_A + \frac{1}{a}x_B, \quad v \equiv |a|p_A - \frac{1}{a}p_B, \]

with \( a \in \mathbb{R}\setminus\{0\} \). [69] showed, that

\[ \rho_{AB} \text{ is separable } \Rightarrow \forall a : (\Delta u)^2 + (\Delta v)^2 \geq a^2 + \frac{1}{a^2}. \]

For Gaussian states, this relation is even a necessary and sufficient criterion for (in)separability. Furthermore, in [70]

\[ \Delta_{EPR}(\rho_{AB}) \equiv \min_a \{1, [(\Delta u)^2 + (\Delta v)^2]/2\} \quad (2.38) \]

was proven to not only qualify entanglement, but also quantify it for symmetric Gaussian states. Because in this case it is a monotonically decreasing function of the entanglement of formation, it serves as an entanglement measure in the above sense. Note, that \( \Delta_{EPR}(\rho_{AB}) = 1 \) iff \( \rho_{AB} \) is separable, and that states with smaller \( \Delta_{EPR} \) are more entangled.

Practically, \( \Delta_{EPR} \) can be computed in two ways. Either by using the standard form coefficients [44] like in [70], or by going to the “usual” EPR-basis (2.19) and adding the respectively smallest eigenvalues of the \( 2 \times 2 \) blockdiagonal submatrices of \( x_+, p_+ \) and \( x_-, p_- \), which were diagonalized separately.

Especially in experiments with atomic ensembles, the EPR-variance is often used [31, 32, 33]. To be precise, there one actually has to consider collective spins in highly polarized states instead of bosonic modes [72].

### 2.5 Quantum State Teleportation

In quantum state teleportation, the fidelity is used as a figure of merit. It is introduced generally in 2.5.1 before in 2.5.2 the idea of teleportation itself is described.

#### 2.5.1 Fidelity

In general, an answer to the question, on how similar two quantum states \( \rho_1 \) and \( \rho_2 \) are, is not obvious and doesn’t appear unambiguously naturally [1]. However, besides the so-called trace distance, the **fidelity** \( F \) is today well established for this task. The latter will be used in this thesis and is defined by

\[ F(\rho_1, \rho_2) \equiv \left( \text{tr}\left\{ \sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}} \right\} \right)^2. \]

If at least one of the states is pure, the fidelity can be seen to be the overlap between the two states, \( F(|\Psi\rangle\langle\Psi|, \rho_2) = \langle \Psi | \rho_2 | \Psi \rangle \). In that case and if \( \rho_1 \) and \( \rho_2 \) are two \( n \)-mode Gaussian states with means \( d_1, d_2 \) and covariance matrices \( \gamma_1, \gamma_2 \), respectively, it can be proved to be [44].
For \( n = 1 \) and one of them, say \( \rho_1 \), being a coherent state, this can be simplified to

\[
F(\rho_1, \rho_2) = \frac{2}{\sqrt{1 + 2V(x_2)^2}} \frac{\sqrt{1 + 2V(p_2)^2}}{\exp^{-\frac{(x_1 - x_2)^2}{1 + 2V(x_2)^2} - \frac{(p_1 - p_2)^2}{1 + 2V(p_2)^2}}.}
\]

(2.39)

### 2.5.2 From Qubit to CV Teleportation

*Quantum state teleportation* is the process of quantum information transfer from a sender to a spatially distant receiver, typically called Alice and Bob, without transmitting any matter. Both parties are equipped with storage devices for quantum states \( A \) and \( B \). The goal is to transmit a quantum state \( |\Psi_{in}\rangle \) of a third storage device \( \Psi \), also located at Alice’s place, to Bob’s, such that \( B \) is in the end in state \( |\Psi_{in}\rangle \). Usually, protocols for this purpose make use of three steps, see Figure 2.2:

i) Entanglement generation between \( A \) and \( B \),

ii) Bell/EPR-measurement of the states in \( A \) and \( \Psi \) at Alice’s side,

iii) Classical communication of the measurement result to Bob and performance of a local operation depending on this result (feedback).

The idea of the protocols can be understood most easily, when \( A \), \( B \) and \( \Psi \) are qubits in an ideal setting like in the original 1993 proposal [9], from where it may be extended to Gaussian CV settings [73, 74, 51]. Here, the first step means preparation of \( A \) and \( B \) in one of the Bell states, say \( |\beta_3\rangle_{AB} \). With their definition (2.21), we can reformulate the product state of the resulting entire system using Bell states for \( \Psi \) and \( A \):

\[
|\Psi_{in}\rangle \Psi |\beta_3\rangle_{AB} = (\alpha|0\rangle + \beta|1\rangle)_\Psi \frac{1}{\sqrt{2}} (|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B)
\]

\[
= \frac{1}{\sqrt{2}} (\alpha|01\rangle_A|0\rangle_B - \alpha|00\rangle_A|1\rangle_B + \beta|11\rangle_A|0\rangle_B - \beta|10\rangle_A|1\rangle_B)
\]

\[
= \frac{1}{2} (|\beta_1\rangle_{AB} (\beta|0\rangle_B - \alpha|1\rangle_B) + |\beta_2\rangle_{AB} (\alpha|0\rangle_B - \beta|1\rangle_B)
\]

\[
+ |\beta_3\rangle_{AB} (-\alpha|0\rangle_B - \beta|1\rangle_B) + |\beta_4\rangle_{AB} (-\beta|0\rangle_B - \alpha|1\rangle_B)).
\]

If now Alice measures \( A \) and \( \Psi \) in the Bell-basis as a second step, with probability 1/4 each, the final state will be a projection onto one of the four states superposed after the last equation sign. \( B \) will then be in one of the according four states, which can, up to an irrelevant global phase, all be transformed to \( |\Psi_{in}\rangle \) by application of
easy local unitaries. So, if Alice classically communicates the measurement result, Bob knows which transformation to apply to end up with \( B \) in state \( |\Psi^m\rangle \). Note, that \( \Psi \) changes its state during this transformation, such that the important no-cloning theorem isn’t violated. Also, because of the need of classical communication to finish the teleportation, information is not transmitted faster than light.

In the CV setting, \( A, B \) and \( \Psi \) are realized by bosonic modes. Instead of in a Bell-state, we first need to prepare \( A \) and \( B \) in a two-mode-squeezed-state entangled as well as possible, i.e. close to an EPR-state. Then for example \( x_- = (x_A - x_B)/\sqrt{2} \) and \( p_+ = (p_A + p_B)/\sqrt{2} \) are squeezed. The second step, the Bell-measurement of \( A \) and \( \Psi \), is for CV replaced by a measurement of \( x'_- = (x_\Psi - x_A)/\sqrt{2} \) and \( p'_+ = (p_\Psi + p_A)/\sqrt{2} \), i.e. by a measurement in the EPR-basis. Using these definitions, Bob’s operators may be rewritten as \( x_B = x_\Psi - \sqrt{2}(x_- + x'_-) \) and \( p_B = p_\Psi + \sqrt{2}(p_+ - p'_+) \). Finally, the measurement result can be classically communicated to Bob, who applies a displacing operation on his variables with gains \( g_1, g_2 \) as described in Appendix C. Typically, one uses \( g_1 = g_2 = 1 \):

\[
\begin{align*}
x_B^{fin} &= x_B + \sqrt{2}g_1x_- = x_\Psi - \sqrt{2}x_- , \\
p_B^{fin} &= p_B + \sqrt{2}g_2p_+ = p_\Psi + \sqrt{2}p_+.
\end{align*}
\]

Since \( x_- \) and \( p_+ \) are squeezed, means and variances of \( B \)'s mode end up to be similar to the ones of \( |\Psi^m\rangle \). For a perfectly entangled EPR-state in the beginning, one has perfect teleportation, and as a figure of merit one can use the fidelity from above. These considerations are described in detail in Chapter 7 in medias res.

As an end of this introduction, two things shall be stressed. Firstly, that, unlike in Chapter 5, the linear feedback is here not used for conditioning, but for the purpose of (optimal) teleportation. And secondly, that the ideal protocols presented in this
introduction rely on the assumption, that one has access to entanglement generation directly between and measurement directly on the systems. This may be true for light setups, but is not the case for atomic ensembles. In Chapter 7, the proposed protocols are therefore more complicated.
3 Interaction of Light with Atomic Ensembles

In this chapter, the off-resonant scattering interaction of light at atomic ensembles is introduced. We consider a spin-polarized ensemble, that interacts with linearly polarized and off-resonant laser light. The dominant effect is population of orthogonally polarized modes, that propagate in the same direction as the laser beam, see Fig. 3.1.

In Section 3.1 we introduce models of the ensemble and of light along the lines of [41] and outline then the derivation of an effective interaction Hamiltonian from first principle quantum optics in [32]. The main point is collectively enhanced coupling of a collective excitation, one bosonic mode, in the atomic ensemble to the bosonic forward-scattered light modes. We follow the derivation of the interaction Hamiltonian of [32, 75] and note, that it is equivalent to use the concept of the polarizability tensor as in [21, 41]. The obtained formulation of the physical system and the interaction makes the connection to the abstract Gaussian studies in Chapters 5 and 6.

In 3.3 then we introduce the input-output formalism at the example of QND-interaction according to [25, 76]. We include a magnetic field in this description and are naturally led to the definition of new bosonic light modes, that will be important in Chapters 4, 6 and 7.

Figure 3.1: A Cs-sample (grey square) is spin-polarized along the \(x\)-direction (black arrow). It interacts with off-resonant laser light (light grey), which propagates along the \(z\)-direction and is linearly polarized in \(y\)-direction. The dominant effect is population of a linearly in \(x\)-direction polarized mode, that also propagates along \(z\) (dark grey).
3.1 Models of the Physical System’s Constituents

3.1.1 Atomic Ensembles

In quantum optics, the term atomic ensemble generally refers to a collection of particles. In general, many physical implementations like room temperature atomic gases, cold atoms in a magneto-optical (MOT) or dipole trap, Bose Einstein Condensates (BECs), atom-like impurities in solid states, Nitrogen-Vacancy (NV) centers in diamond or ultracold atoms in optical lattices have been considered for this task, each of them having different advantages and disadvantages.

This thesis deals with room-temperature ensembles of \( N_A \approx 10^{12} \) atoms, that are implemented as atomic vapor in coated glass cells with a size on the order of \( \text{cm}^3 \). In experiments, usually alkali metals like the well-studied Cesium (Cs) are employed, while the basic theoretical model grasps the main properties of any Hydrogen-like atoms with just one electron in the valence shell. In the thesis at hand, for the interaction with light we use the \(^{133}\text{Cs}-D_2\) transition \( ^6S_{1/2}(F = 4) \rightarrow ^6P_{3/2}(F' = 3, 4, 5) \), where \( F \) is the atom’s total angular momentum. Hyperfine splitting leads to a complex multi-level structure, see Fig. 3.2a. For the excited state manifold \( \{|F, m_F\rangle | F' = 3, 4, 5, -F' \leq m_F \leq F' \} \) it is included in our model, whereas the existence of other states of the ground state manifold \( \{|F, m_F\rangle | F = 4, m_F = -4, ..., 4 \} \) besides the two relevant with \( m_F = 3, 4 \) leads to losses in our model.

Initially, the sample is assumed to be fully spin-polarized in \( x \)-direction, such that all atoms are in the state \( |F = 4, m_F = 4 \rangle \). Then, \( \langle J_x \rangle = 4N_A \) and \( (\Delta J_y)^2 = (\Delta J_z)^2 = 2N_A \) for the ensemble’s collective angular momentum \( J = \sum_{i=1}^{N_A} F_i \), where \( F_i \) is the total angular momentum of the \( i^{th} \) atom. This state is an eigenstate of \( J_x \) with maximal eigenvalue \( j = 4N_A \). Note, that in this thesis the quantization is along the \( x \)- and not along the \( z \)-axis as in many textbooks. In experiments the state can be prepared by optical pumping.

Deviations from this state are expressed by collective ladder operators \( J^\pm = J_y \pm i J_z \). In the Holstein-Primakoff transformation \([77, 78]\), they are given with the annihilation and creation operators of a fictitious bosonic mode, \([b, b^\dagger] = 1\), by

\[
J^+ = \sqrt{N_A} b \sqrt{1 - b^\dagger b / 2j}, \quad J^- = \sqrt{N_A} b \sqrt{1 - b^\dagger b / 2j}.
\]

Identifying \( J_x = j - b^\dagger b \) due to

\[
J_x^2 = J^2 - J_y^2 - J_z^2 = j(j + 1) - \frac{1}{2}(J^+ J^- + J^- J^+) = (j - b^\dagger b)^2,
\]

the commutation relation \([J_+, J_-] = -2J_x\) is satisfied. The initial state corresponds to the vacuum state of the bosonic mode and is called a coherent spin state (CSS). For small deviations like they appear in experiments we have \( \langle b^\dagger b \rangle \ll 2j \), and one can approximate \( J_+ \approx \sqrt{2j} b \) and \( J_- \approx \sqrt{2j} b^\dagger \). Therefore
Figure 3.2: (a) Relevant manifolds of the Cs-multi-level structure for the $D_2$-transition. The atoms are all initially prepared in the ground state $|g⟩ = |F = 4, m_F = 4⟩$. The incident laser of frequency $ω_c$ is detuned with respect to that transition. (b) An exemplary set of transitions between ground and excited manifold. The $y$-polarized laser drives the cross-transitions (solid arrows), and vertical transitions involve $x$-polarized photons (dashed arrows). We choose the energy of $|g⟩$ as zero, $E = 0$.

$$x_A ≈ J_y / \sqrt{j}, \quad p_A ≈ J_z / \sqrt{j}$$

(3.1)

for the atomic operators $x_A = \frac{1}{\sqrt{2}} (b + b^\dagger)$ and $p_A = \frac{1}{\sqrt{2}i} (b - b^\dagger)$. This is called the Holstein-Primakoff approximation. The new operators fulfill the commutation relation $[x_A, p_A] ≈ [J_y, J_z]/j = i J_x/j ≈ i$.

Because we have characterized the ensemble now with a bosonic mode, we call a state a spin squeezed state (SSS) in analogy to the CSS, if $(Δx_A)^2 < 2N_A$ or $(Δp_A)^2 < 2N_A$, compare Section 2.2.4.

3.1.2 Light

A major observation is, that light scattering in forward direction is collectively enhanced [32]. We treat scattering in other directions as noise, and a one-dimensional model along the propagation direction of the incident laser covers the relevant physics. We choose this one dimension to be the z-axis and label its coordinate by $r$.

The incident laser is in linear $y$-polarization, and the collectively enhanced process leads to scattered light, which is linearly polarized in $x$-direction, see Fig. 3.1. Other light modes are treated as noise. The laser as a strong coherent field is then given by the central mode with wave number $k_c$, according central frequency $ω_c$ and a narrow bandwidth $b$. The light modes of wave number $k$ and polarization along $x$ that are populated during the scattering are given by

$$H_L = \int d_k \hbar ω_k a_{k,x}^\dagger a_{k,x},$$

(3.2)
where the creation and annihilation operators fulfill the usual commutation relation
\[ [a_{k,x}, a_{k',x}^\dagger] = \delta(k - k'). \]

**Continuous Spatial Modes** We are naturally led to the use of spatially localized bosonic modes in the following. They are defined via
\[
\begin{align*}
    a_x(r) &\equiv \frac{1}{\sqrt{2\pi}} \int_b dk e^{i(k-k_c)r} a_{k,x}, \\
    a_x^\dagger(r) &\equiv \frac{1}{\sqrt{2\pi}} \int_b dk e^{-i(k-k_c)r} a_{k,x}^\dagger,
\end{align*}
\]
(3.3)
where \( \int_b \) means integration over \( k_c - b \leq k \leq k_c + b \). They fulfill the commutation relation
\[
[a_x(r), a_x^\dagger(r')] = \frac{1}{2\pi} \int_b dk \int_b dk' [a_{k,x}, a_{k',x}] e^{i(k-k_c)r} e^{-i(k'-k_c)r'}
\]
\[
= \frac{1}{2\pi} \int_b dk e^{i(k-k_c)(r-r')} \delta_b(r-r').
\]
\( \delta_b(r-r') \) can be seen as a delta-function of width \( b \), which is centered at \( k_c \).

These definitions give rise to bosonic light quadratures, that are used in the following chapters:
\[
\begin{align*}
    x_L(r) &\equiv \frac{1}{\sqrt{2}} [a_x(r) + a_x^\dagger(r)], \\
    p_L(r) &\equiv -\frac{i}{\sqrt{2}} [a_x(r) - a_x^\dagger(r)].
\end{align*}
\]
(3.4)
If the laser now is in \( x- \) instead of in \( y- \)polarization, we need to look at light quadratures
\[
\begin{align*}
    x_{L,y}(r) &\equiv \frac{1}{\sqrt{2}} [a_y(r) + a_y^\dagger(r)], \\
    p_{L,y}(r) &\equiv -\frac{i}{\sqrt{2}} [a_y(r) - a_y^\dagger(r)].
\end{align*}
\]
(3.5)
instead, where the creation and annihilation operators are defined analogously to (3.3).

**Discrete Spatial Modes** The continuously quantized radiation field can equivalently be described by a discretely quantized one \[79\]. Let the quadratures \( x_{L,n} \) and \( p_{L,n} \) describe the mode at \( [r,r + dr] \). Because of \( dr\delta(r-r') \to \delta_{n,n'} \), then we obtain \( [a_{L,n}, a_{L,n'}^{\dagger}] = \delta_{n,n'} \) just for \( \sqrt{dr}a_L(r)^{(l)} \to a_{L,n}^{(l)} \) or, equivalently,
\[
\begin{align*}
    \sqrt{dr}x_L(r) &\to x_{L,n}, \\
    \sqrt{dr}p_L(r) &\to p_{L,n}.
\end{align*}
\]
Since \( dr = c \, dt \), the discretization therefore induces a rescaling of the time in the derivation of the differential equation in Chapter 6.
3.2 The Effective Interaction Hamiltonian

We start with a minimal coupling Hamiltonian of an ensemble of effective one-electron atoms to the electromagnetic light field and apply the dipole approximation [60]. The total Hamiltonian is given by a light contribution, an atomic contribution and an interaction part. We focus on the interaction Hamiltonian first and take care of the other ones in the end. As emphasized before, light scattering in forward direction is collectively enhanced and we just consider a one-dimensional theory [32]. The incident laser with wave number \( k_c \) is assumed to be linearly polarized in \( y \)-direction and to off-resonantly drive the \( D_2 \)-transition.

We encode our protocols in the ground state \(| g \rangle \equiv |F = 4, m_F = 4 \rangle\), a metastable state \(| s \rangle \equiv |F = 4, m_F = 3 \rangle\) and in the states \(|F', m_{F'} \rangle |F' = 3, 4, 5; 3 \leq m_{F'} \leq 4 \rangle\) from the excited state manifold. Because we start with a highly populated ensemble, in which nearly all atoms are in the ground state, states \(|F = 4, m_F < 3 \rangle\) are almost not populated and can be treated as noise. We pick two of the excited states, to which transitions are allowed by the rules of angular momentum conservation, as an example and label them \(|e_g \rangle\) and \(|e_s \rangle\). The energy of \(|g \rangle\) is chosen as the zero energy level, see Fig. 3.2b. In the rotating wave approximation, then the incident laser field drives the transitions \(|g \rangle \rightarrow |e_s \rangle\) and \(|s \rangle \rightarrow |e_g \rangle\), whereas coupling to \(x\)-polarized modes of the electromagnetic field gives rise to transitions \(|e_g \rangle \rightarrow |g \rangle\) and \(|e_s \rangle \rightarrow |s \rangle\).

By going to a rotating frame and adiabatically eliminating the barely populated excited states \(|e_g \rangle\) and \(|e_s \rangle\), we get the interaction Hamiltonian

\[
H_{\text{int}} = \int dk g(k) \{ \mu \sum_{i=1}^{N} \sigma_i^- e^{i(k_c-k)r_i} + \nu \sum_{j=1}^{N} \sigma_j^+ e^{i(k_c-k)r_j} \} a_{k,x}^\dagger + \text{h.c.}.
\]

The coupling is here factorized into the transition specific \( \mu/\nu \) and the for all transitions constant \( g(k) \). We impose the normalization condition \( \mu^2 - \nu^2 = 1 \). Each allowed transition to all possible excited states \(|e_g \rangle\) and \(|e_s \rangle\) contributes in the same way: solely to the total \( \mu \) and \( \nu \). Also, for each atom \( i \), a ladder operator \( \sigma_i^+ \equiv |s_i \rangle \langle g| \) has been introduced. It gives rise to excitations, i.e. transitions from the ground into the metastable state, \(|g_i \rangle \rightarrow |s_i \rangle\). Its Hermitian conjugate is given by \( \sigma_i^- \equiv |g_i \rangle \langle s| \) and accounts for the opposite transition \(|s_i \rangle \rightarrow |g_i \rangle\). Note, that an excitation in this sense has nothing to do with the adiabatically eliminated excited states \(|e_g \rangle\) and \(|e_s \rangle\).

Next, we make the approximation of a pointwise atomic ensemble and set \( r_i = r_0 \) for all atoms \( i \). With this, we can identify the collective operator \( \frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} \sigma_j^+ \) as the symmetric term of a Fourier transformation of all single-atom creation operators. It leads the excitation of a symmetric spin wave:

\[
\frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} \sigma_j^+ |g_1, \ldots, g_{N_A} \rangle = \frac{1}{\sqrt{N_A}} \sum_{j=1}^{N_A} |g_1, \ldots, g_{j-1}, s_j, g_{j+1}, \ldots, g_{N_A} \rangle.
\]

It is exactly this symmetric mode, to which the forward scattered light couples through constructive interference in the collectively enhanced process.
We assume the incident laser to have narrow bandwidth \( b \), so that in our case of (elastic) Rayleigh-scattering also the involved radiation field modes differ in frequency at most by \( b \). In the corresponding small interval of wave numbers \( k_c - b \leq k \leq k_c + b \), the coupling is nearly constant, \( g(k) \approx g \):

\[
H_{\text{int}} = g \int_b dk \left\{ \mu \sum_{i=1}^N \sigma_i^- + \nu \sum_{j=1}^N \sigma_j^+ \right\} e^{i(k_c-k)r_0} a_{k,x}^\dagger + h.c.
\]

\[
= g \{ \mu J^- + \nu J^+ \} \int_b dk e^{i(k_c-k)r_0} a_{k,x}^\dagger + h.c.
\]

In the last equality, we defined the collective ladder operators \( J^- \equiv \sum_{i=1}^N \sigma_i^- \) and \( J^+ \equiv \sum_{j=1}^N \sigma_j^+ \). That they approximately coincide with the ladder operators for the total angular momentum of the ensemble is a consequence of the high polarization [80]. The integral is now exactly the spatial bosonic light mode at \( r_0 \), that we defined in Section 3.1.2. Always just the light mode at the location of the ensemble participates in the interaction. With \( J^\pm = J_y \pm iJ_z \), we can rewrite the Hamiltonian as

\[
H_{\text{int}} = g \sqrt{\pi} \{ Z J_y x_L(0) + \frac{1}{Z} J_z p_L(0) \}.
\]

Here, we also made use of the definition (3.4) for the light quadratures and set \( r_0 = 0 \). We defined \( Z \equiv \mu + \nu \) and \( Z^{-1} = \mu - \nu \), which suits the normalization condition. In the Holstein-Primakoff-approximation, (3.1), this Hamiltonian can now be formulated as an interaction between just one bosonic atomic and one bosonic light mode:

\[
H_{\text{int}} = \frac{\hbar \kappa}{\sqrt{T}} \{ Z A x_L(0) + \frac{1}{Z} p_{AP L}(0) \} = \frac{\hbar \kappa}{\sqrt{T}} \{ \mu H_{bs} + \nu H_{tms} \}.
\] (3.6)

\( T \) is the duration of the laser pulse and has been separated for accordance with the literature [11] form the dimensionless coupling constant \( \kappa \). In this derivation, \( \kappa \) can be calculated from the first principle coupling constant, and in experiments it is typically of the order of unity. The second form of the Hamiltonian shows, that \( \mu \) and \( \nu \) are the weights of a passive beam splitter \( H_{bs} = x_A x_L + p_{AP L} \) and an active two-mode-squeezer/parametric gain \( H_{tms} = x_A x_L - p_{AP L} \) contribution.

These weights depend on the detuning, and therefore the interaction Hamiltonian can be designed. In the following chapters, we use basically two specific ones: one we term the realistic Hamiltonian, and the other is the well-known QND-/Faraday-Hamiltonian. The realistic is a typical Hamiltonian used in recent experiments, for which we calculate in Appendix A: \( \mu = 1.45 \) and \( \nu = 1.05 \), i.e. \( Z = 2.5 \) and \( 1/Z = 0.4 \). The QND-/Faraday-Hamiltonian is obtained for \( Z \to \infty \) while keeping the factor \( \frac{\hbar \kappa}{\sqrt{T}} \) constant, i.e. \( H_{\text{int}} \propto x_A x_L \). Physically, the QND-Hamiltonian is achieved in the far off-resonant limit.
All together, we can describe the system by
\[ H = H_L + H_{int} \]

where \( H_L = \int dk \hbar (\omega_k - \omega_c) a_{k,x}^\dagger a_{k,x} \) is the light Hamiltonian in the rotating frame, and because of the choice of the energy’s zero level and the adiabatic elimination of the excited states, we do not have any purely atomic contribution in our model. So, finally

\[ H = H_L + H_{int} = \int dk \hbar (\omega_k - \omega_c) a_{k,x}^\dagger a_{k,x} + \frac{\hbar \kappa}{\sqrt{T}} \left\{ Z x_A x_L(0) + \frac{1}{Z} p_A p_L(0) \right\}. \]  

(3.7)

The whole derivation can be conducted analogously for a laser with linear \( x \)- instead of \( y \)-polarization, see Fig. 3.3. Then the scattered modes of the radiation field are those with \( y \)-polarization in (3.5). For example, the QND-coupling reads

\[ H_{int} = \frac{\hbar \kappa}{\sqrt{T}} p_{AP_{L,y}}(0) \]

(3.8)

in this configuration instead of \( H_{int} \propto x_A x_L(0) \) like above. \( Z \) has been absorbed in \( \kappa \) in this notation. Note, that one form can be transformed into the other by a local redefinition of operators. We consider both polarization cases in this thesis and leave out the polarization index for the light quadratures, as the polarization should always be clear from the context.

### 3.3 Input-Output-Relations for QND-Interaction

We consider QND-interaction between an in \( x \)-direction spin-polarized atomic ensemble and a linearly in \( x \)-direction polarized laser beam. In addition, a homogeneous magnetic field is applied to the ensemble in \( -x \)-direction. We obtain a set of input-output-relations for this setup by solving the Heisenberg equations of motion and by deriving the relations without a magnetic field first.

The magnetic field leads to Larmor precession of the total angular momentum of the ensemble with frequency \( \Omega \): \( H_{mag} = -\hbar \Omega J_x \). In the Holstein-Primakoff approximation, we have

\[ H_{mag} = -\hbar \Omega (j - b^\dagger b) \approx -\hbar \Omega j + \frac{\hbar \Omega}{2} (x_A^2 + p_A^2) - \frac{1}{2} \hbar \Omega \]

(3.9)

Since the first and the third term as numbers just give rise to global phases, we can leave them out and describe the magnetic Hamiltonian as

\[ H_{mag} = \hbar \Omega (x_A^2 + p_A^2)/2. \]

Treating the light modes continuously, we have, according to (3.7) and (3.8),

\[ H = H_{mag} + H_L + H_{int} = \hbar \Omega (x_A^2 + p_A^2)/2 + \int dk \hbar (\omega_k - \omega_c) a_{k,x}^\dagger a_{k,x} + \frac{\hbar \kappa}{\sqrt{T}} p_{AP_L}(0). \]
3.3 Heisenberg Equations

First, the Heisenberg equations are solved explicitly, and after the definition of effective modes a $S$-matrix in the sense of 2.3.1 can be read off for them. For atomic variables, we have

$$\partial_t x_A(t) = i \frac{\hbar}{\hbar} [H, x_A(t)] = \Omega p_A(t) + \frac{\kappa}{\sqrt{T}} p_L(0, t),$$
$$\partial_t p_A(t) = i \frac{\hbar}{\hbar} [H, p_A(t)] = -\Omega x_A(t).$$

For the light modes, we first evaluate the equations of motion for the annihilation operators due to the free field Hamiltonian $H_L$,

$$\partial_t a_y(r, t) = i \frac{\hbar}{\hbar} [H_L, a_y(r, t)] = i \frac{\hbar}{\hbar} \int dk \hbar (\omega_k - \omega_c) \frac{1}{\sqrt{2\pi}} \int dk' [a_{k+y}^+, a_{k', y}, a_{k', y}^+] e^{i(k' - k_c)r}$$
$$= -i \sqrt{\frac{2\pi}{2}} \int dk(k - k_c) c a_{k, y} e^{i(k - k_c)r} = -c \partial_r a_y(r, t).$$

This is, as expected, the wave equation, and therefore e. g. $\partial_r x_L(r, t) = -c \partial_r x_L(r, t)$. So, the whole evolution of the light modes is given by

$$(\partial_t + c \partial_r)x_L(r, t) = i \frac{\hbar}{\hbar} [H_{int}, x_L(r, t)] = \frac{\kappa c}{\sqrt{T}} p_A(t) \delta(r),$$
$$(\partial_t + c \partial_r)p_L(r, t) = i \frac{\hbar}{\hbar} [H_{int}, p_L(r, t)] = 0.$$
Now, variables are transformed according to $u : \mathbb{R}^2 \to \mathbb{R}^2, u((r,t)) = (\xi, \tau) = (ct - r, t)$, and light modes in new variables denoted by a bar: $\bar{x}_L(\xi, \tau) = x_L(r, t)$. Then, according to the chain rule for $x_L : \mathbb{R}^2 \to \mathbb{R}$ and $u^{-1} : \mathbb{R}^2 \to \mathbb{R}^2, u^{-1}((\xi, \tau)) = (r, t) = (c\tau - \xi, \tau),$

$$\partial_r \bar{x}_L(\xi, \tau) = \partial_r x_L(u^{-1}(\xi, \tau)) = \frac{dx_L}{du^{-1}} \frac{du^{-1}}{dT} \left( \partial_{u_1} x_L \partial_{u_2} x_L \right) \left( \partial_{u_1} u^{-1} \partial_{u_2} u^{-1} \right)$$

$$= \partial_r x_L * c + \partial_t x_L * 1 = (\partial_l + c\partial_r)x_L(r, t).$$

The new coordinate frame is fixed on the light pulse and labels the modes starting with the one which enters the atomic ensemble first. For ease of notation, we write $t$ for $\tau$ in the following.

### 3.3.2 Input-Output-Relations without magnetic field

Now, we turn to the case $\Omega = 0$, and in the new variables, we have

$$\partial_t x_A(t) = \frac{\kappa}{\sqrt{T}} \bar{p}_L(ct, t), \quad \partial_t p_A(t) = 0,$$

$$\partial_t \bar{x}_L(\xi, t) = \frac{\kappa c}{\sqrt{T}} p_A(t) \delta(c\tau - \xi), \quad \partial_t \bar{p}_L(\xi, t) = 0.$$

Because $p_A$ and $p_L(\xi)$ are temporally conserved quantities, these differential equations are solved by

$$x_A(t) = x_A(0) + \frac{\kappa}{\sqrt{T}} \int_0^t d\tau \bar{p}_L(ct, 0),$$

$$p_A(t) = p_A(0),$$

$$\bar{x}_L(\xi, t) = \bar{x}_L(\xi, 0) + \frac{\kappa}{\sqrt{T}} p_A(0),$$

$$\bar{p}_L(\xi, t) = \bar{p}_L(\xi, 0).$$

Generally, the idea of input-output-relations is to define effective modes before and after the whole interaction of time $T$ and to derive relations between them.

Here, for atomic variables, we naturally set $x_A^{in} \equiv x_A(0), p_A^{in} \equiv p_A(0), x_A^{out} \equiv x_A(T)$ and $p_A^{out} \equiv p_A(T)$. For light, define integrated modes over the whole pulse

$$x_L^{in} \equiv \frac{1}{\sqrt{T}} \int_0^T d\tau \bar{x}_L(ct, 0), \quad p_L^{in} \equiv \frac{1}{\sqrt{T}} \int_0^T d\tau \bar{p}_L(ct, 0),$$

$$x_L^{out} \equiv \frac{1}{\sqrt{T}} \int_0^T d\tau \bar{x}_L(ct, T), \quad p_L^{out} \equiv \frac{1}{\sqrt{T}} \int_0^T d\tau \bar{p}_L(ct, T).$$
It is easily calculated, that the chosen normalization guarantees standard bosonic commutation relations like \([x_L^{in}, p_L^{in}] = i\) for the new modes. With the above, the set of input-output-relations can be calculated to be

\[
\begin{align*}
x^{out}_A &= x^{in}_A + \kappa p^{in}_L, \\
p^{out}_A &= p^{in}_A, \\
x^{out}_L &= \frac{1}{\sqrt{T}} \int_0^T d\tau \bar{x}_L(ct, T) = \frac{1}{\sqrt{T}} \int_0^T d\tau \left( \bar{x}_L(ct, 0) + \frac{\kappa}{\sqrt{T}} p_A(0) \right) = x^{in}_L + \kappa p^{in}_A, \\
p^{out}_L &= \frac{1}{\sqrt{T}} \int_0^T d\tau \bar{p}_L(ct, T) = \frac{1}{\sqrt{T}} \int_0^T d\tau \bar{p}_L(ct, 0) = p^{in}_L.
\end{align*}
\]

As the state of the \(p\)-variables is mapped on the \(x\)-variables, while the \(p\)-variables do not change during this process, quantum non-demolition (QND) measurements may be performed this way.

### 3.3.3 Input-Output-Relations with a homogeneous magnetic field

With a homogeneous magnetic field, in transformed light variables we have

\[
\begin{align*}
\partial_t x_A(t) &= \Omega p_A(t) + \frac{\kappa}{\sqrt{T}} \bar{p}_L(ct, t), \\
\partial_t p_A(t) &= -\Omega x_A(t), \\
\partial_t \bar{x}_L(\xi, t) &= \frac{\kappa c}{\sqrt{T}} p_A(t) \delta(ct - \xi), \\
\partial_t \bar{p}_L(\xi, t) &= 0.
\end{align*}
\]

The equations can be solved conveniently in a rotating frame with respect to \(H_{mag}\), for which we define new atomic variables

\[
\begin{pmatrix}
\tilde{x}_A(t) \\
\tilde{p}_A(t)
\end{pmatrix} = \begin{pmatrix}
\cos(\Omega t) & -\sin(\Omega t) \\
\sin(\Omega t) & \cos(\Omega t)
\end{pmatrix} \begin{pmatrix}
x_A(t) \\
p_A(t)
\end{pmatrix}
\]

and calculate the set of differential equations in these variables

\[
\begin{align*}
\partial_t \tilde{x}_A(t) &= \cos(\Omega t) \partial_t x_A(t) - \Omega \sin(\Omega t) x_A(t) - \sin(\Omega t) \partial_t p_A(t) - \Omega \cos(\Omega t) p_A(t) \\
&= \cos(\Omega t) \left( \Omega p_A(t) + \frac{\kappa}{\sqrt{T}} \bar{p}_L(ct, t) \right) - \Omega \sin(\Omega t) x_A(t) + \sin(\Omega t) \Omega x_A(t) \\
&\quad - \Omega \cos(\Omega t) p_A(t) \\
&= \cos(\Omega t) \frac{\kappa}{\sqrt{T}} \bar{p}_L(ct, t),
\end{align*}
\]
\[ \partial_t \bar{p}_A(t) = \sin(\Omega t) \partial_t x_A(t) + \Omega \cos(\Omega t) x_A(t) + \cos(\Omega t) \partial_t p_A(t) - \Omega \sin(\Omega t) p_A(t) \]

\[ = \sin(\Omega t) \left( \Omega p_A(t) + \frac{\kappa}{\sqrt{T}} \bar{p}_L(ct, t) \right) + \Omega \cos(\Omega t) x_A(t) - \cos(\Omega t) \Omega x_A(t) \]

\[ - \Omega \sin(\Omega t) p_A(t) = \sin(\Omega t) \frac{\kappa}{\sqrt{T}} \bar{p}_L(ct, t), \]

\[ \partial_t x_L(\xi, t) = \frac{\kappa c}{\sqrt{T}} \left( - \sin(\Omega t) x_A(t) + \cos(\Omega t) p_A(t) \right) \delta(ct - \xi), \]

\[ \partial_t \bar{p}_L(\xi, t) = 0. \]

Contrary to the case without any magnetic field, already here it can be seen, that now \( \bar{p}_L \) affects both \( \bar{x}_A \) and \( \bar{p}_A \), and in turn also both influence \( \bar{x}_L \). This can be understood, as physically always only the component of the total angular momentum in \( z \)-direction contributes to \( \bar{p}_L \). By rotating \( x_A \) and \( p_A \) in the \( y-z \)-plane with the help of the magnetic field, both operators then do so. Also, at each point of time it is the projection of the atomic operators onto the \( z \)-axis, that has an effect on \( \bar{x}_L \).

Because \( \bar{p}_L(\xi) \) is conserved, the set of differential equations is solved by

\[
\begin{align*}
\bar{x}_A(t) &= \bar{x}_A(0) + \frac{\kappa}{\sqrt{T}} \int_0^t d\tau \cos(\Omega \tau) \bar{p}_L(\tau, 0), \\
\bar{p}_A(t) &= \bar{p}_A(0) + \frac{\kappa}{\sqrt{T}} \int_0^t d\tau \sin(\Omega \tau) \bar{p}_L(\tau, 0), \\
\bar{x}_L(\xi, t) &= \bar{x}_L(\xi, 0) + \frac{\kappa}{\sqrt{T}} \left[ - \sin \left( \frac{\Omega \xi}{c} \right) \bar{x}_A \left( \frac{\xi}{c} \right) + \cos \left( \frac{\Omega \xi}{c} \right) \bar{p}_A \left( \frac{\xi}{c} \right) \right], \\
\bar{p}_L(\xi, t) &= \bar{p}_L(\xi, 0). \tag{3.10}
\end{align*}
\]

Like before, at this point we define proper input-output-modes and derive relations between them. For atomic variables, we can make the same choice as in the case without any magnetic field, \( x_A^{in} = x_A(0), p_A^{in} = p_A(0), x_A^{out} = x_A(T) \) and \( p_A^{out} = p_A(T) \), but for the light modes it is more complicated now:

\[
\begin{pmatrix} x_s^{in} \\ x_s^{out} \end{pmatrix} \equiv \sqrt{\frac{2}{T}} \int_0^T d\tau \sin(\Omega \tau) \begin{pmatrix} \bar{x}_L(\tau, 0) \\ \bar{x}_L(\tau, T) \end{pmatrix}, \quad \begin{pmatrix} p_s^{in} \\ p_s^{out} \end{pmatrix} \equiv \sqrt{\frac{2}{T}} \int_0^T d\tau \sin(\Omega \tau) \begin{pmatrix} \bar{p}_L(\tau, 0) \\ \bar{p}_L(\tau, T) \end{pmatrix}.
\]

Analogously, by replacing \( \sin \) by \( \cos \) in all definitions, \( x_e^{in}, x_e^{out}, p_e^{in} \) and \( p_e^{out} \) are defined. The normalizations are chosen such that the light modes are asymptotically canonical, e. g.

\[
[x_s^{in}, p_s^{in}] = \frac{2i}{T} \int_0^T d\tau \sin^2(\Omega \tau) \approx i,
\]
and independent, \([x_{c,1}^{in}, p_{c,1}^{in}] \approx 0\). This is, because approximations due to \(\int_0^T d\tau \sin^2(\Omega \tau) = \int_0^T d\tau \cos^2(\Omega \tau) = T/2 + O((\Omega T)^{-1})\) and \(\int_0^T d\tau \sin(\Omega \tau) \cos(\Omega \tau) = O((\Omega T)^{-1})\) can be used in the regime \((\Omega T)^{-1} \ll 1\), which is typical for experiments with \(T \approx ms\) and \(\Omega \approx MHz\).

For these mode definitions, we have

\[
x_{A}^{out} = x_{A}^{in} + \frac{\kappa}{\sqrt{2}} p_{c}^{in}, \\
p_{A}^{out} = p_{A}^{in} + \frac{\kappa}{\sqrt{2}} p_{s}^{in}, \\
x_{s}^{out} = p_{s}^{in}, \\
p_{c}^{out} = p_{c}^{in}.
\]

Also in the regime \((\Omega T)^{-1} \ll 1\), it can be calculated with (3.10), that

\[
x_{s}^{out} = \sqrt{\frac{2}{T}} \int_0^T d\tau \sin(\Omega \tau) \left( \bar{x}_L(ct, 0) + \frac{\kappa}{\sqrt{T}} \left( -\sin(\Omega \tau) \bar{x}_A(\tau) + \cos(\Omega \tau) \bar{p}_A(\tau) \right) \right)
\approx x_{s}^{in} - \frac{\kappa}{\sqrt{2}} x_{A}^{in} - \frac{\sqrt{2} \kappa^2}{T^{3/2}} \int_0^T d\tau' \int_0^\tau d\tau'' \sin^2(\Omega \tau') \cos(\Omega \tau') \bar{p}_L(\tau', 0)
+ \sin(\Omega \tau) \cos(\Omega \tau) \sin(\Omega \tau') \bar{p}_L(\tau', 0)
\approx x_{s}^{in} - \frac{\kappa}{\sqrt{2}} x_{A}^{in} - \frac{\sqrt{2} \kappa^2}{T^{3/2}} \int_0^T d\tau' \frac{T - \tau'}{2} \cos(\Omega \tau') \bar{p}_L(\tau', 0),
\]

(3.11)

where the integral over \(\tau\) can be performed by interchanging \(\int_0^T d\tau \int_0^\tau d\tau' \rightarrow \int_0^T d\tau' \int_0^T d\tau\). The last term can be understood as back action of light onto itself, as according to (3.10) every slice \(\xi\) of the light pulse is changed by atomic variables at time \(\xi/c\), which in turn carry the integrated signal of all slices up to \(\xi\). This term can be expressed as a sum of a term proportional to \(p_{c}^{in}\) and one to \(p_{s}^{in}\), as part of a new mode, with \(x_{c,1}^{in}\) defined analogously by replacing \(\bar{p}_L\) by \(\bar{x}_L\). The new mode is again approximately canonical, \([x_{c,1}^{in}, p_{c,1}^{in}] \approx i\), and independent of all other modes, e.g. \([x_{c}^{in}, p_{c}^{in}] \approx 0\). Similar calculations and definition of a mode with \(x_{s,1}^{in}\) and \(p_{s,1}^{in}\) by replacing \(\cos\) by \(\sin\) in (3.12) finally leads to

\[
x_{s}^{out} = x_{s}^{in} - \frac{\kappa}{\sqrt{2}} x_{A}^{in} - \frac{\kappa^2}{4} p_{c}^{in} - \frac{\kappa^2}{4 \sqrt{3}} p_{c,1}^{in}, \\
x_{c}^{out} = x_{c}^{in} + \frac{\kappa}{\sqrt{2}} p_{A}^{in} + \frac{\kappa^2}{4} p_{s}^{in} + \frac{\kappa^2}{4 \sqrt{3}} p_{s,1}^{in}.
\]

(3.13)
Two things are worth mentioning in the end. First, it is possible to derive input-output-relations for the new modes as well, which then involve again other independent and canonical light modes with \((x/p)^{\pm n}_{(c/s)}\). Deriving in turn output equations for them etc. yields a hierarchy of input-output-relations for independent bosonic modes.

Second, the utter utility of the defined sine- and cosine-modes shall be stressed. In experiments, the modulation of output light with \(\cos(\Omega t)\) and \(\sin(\Omega t)\) is very useful to drastically reduce technical noise, because the new modes are linear combinations of lower- and upper-sideband modes \cite{21}.
4 Entanglement Generation: State of the Art and Outline

The idea to entangle two atomic ensembles by means of interaction with light has been introduced in 2000 [30]. There it has been suggested to let one laser beam interact with two atomic ensembles one after the other in the far off-resonant limit, i.e. with QND-/Faraday-interaction. In 2001, an experimental demonstration with room-temperature Cs-ensembles was successful [31]. As this idea is fundamental to the thesis at hand and as the calculations are the basis Chapter 7, we summarize it briefly in 4.1. Readers familiar with the subject may skip this technical introductory section. In 4.2 we present further developments of this protocol and stress components important in the following. Section 4.3 then picks up the discussion and introduces the major strategies of this thesis to improve the protocol under investigation. These approaches are worked out in Chapters 5 and 6 in detail, and in Chapter 7 the results are finally applied to teleportation protocols.

4.1 The Original Idea

We consider two atomic ensembles, both spin-polarized in \( x \)-direction, which undergo a QND-interaction with a light beam linearly polarized in \( x \)-direction, see Fig. 4.1. The two ensembles are located at \( r = 0 \) and \( r = l \), and their bosonic variables are denoted with subscripts \( A1 \) and \( A2 \), respectively. Magnetic fields of equal strength \( \Omega \) and opposite orientation are applied to them, which leads to different signs in \( H_{\text{mag}} \).

With (3.7) and (3.9) we get:

\[
H = H_{\text{mag}} + H_{L} + H_{\text{int},1} + H_{\text{int},2},
\]

\[
H_{\text{mag}} = -\frac{\hbar\Omega}{2}(x_{A1}^{2} + p_{A1}^{2}) + \frac{\hbar\Omega}{2}(x_{A2}^{2} + p_{A2}^{2}),
\]

\[
H_{L} = \int d\omega \hbar(\omega - \omega_{c})a_{y,\omega}^{\dagger}a_{y,\omega},
\]

\[
H_{\text{int},1} = \frac{\hbar k}{\sqrt{T}}p_{A1}p_{L}(0), \quad H_{\text{int},2} = \frac{\hbar k}{\sqrt{T}}p_{A2}p_{L}(l).
\]

If we assume an ideal treatment like in Section 3.3, meaning that the light field propagates between the two ensembles without any losses, due to (3.10) \( p_{L}(\xi, t) \) is a preserved quantity during the separate interaction with each ensemble. Because
the atomic variables are only influenced by this preserved quantity, and because the change in $x_{c/s}$ is realized by just adding a term as shown in (3.13), we may set $l = 0$ in the following.

In the EPR-basis (2.19), the Hamilton operator may then be rewritten as

$$H = -\hbar \Omega (x_+ x_- + p_+ p_-) + H_L + \hbar \kappa \sqrt{\frac{2}{T}} p_L(0).$$

From now on, we basically follow the concepts and calculations of the previous chapter. Following the calculations in 3.3, the Heisenberg equations are given by

$$\partial_t x_+(t) = -\Omega p_-(t) + \sqrt{\frac{2}{T}} \kappa \bar{p}_L(ct, t), \quad \partial_t p_+(t) = \Omega x_-(t),$$
$$\partial_t p_-(t) = \Omega x_+(t), \quad \partial_t x_-(t) = -\Omega p_+(t),$$
$$\partial_t \bar{x}_L(\xi, t) = \sqrt{\frac{2}{T}} \kappa p_+(t) \delta(ct - \xi), \quad \partial_t \bar{p}_L(\xi, t) = 0.$$

Then, we go to an interaction picture with respect to $H_A = -\hbar \Omega (x_+ x_- + p_+ p_-)$ by an operator transformation

$$\begin{pmatrix}
\bar{x}_+(t) \\
\bar{p}_+(t) \\
\bar{x}_-(t) \\
\bar{p}_-(t)
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos(\Omega t) & 0 & 0 & \sin(\Omega t) \\
0 & \cos(\Omega t) & -\sin(\Omega t) & 0 \\
0 & \sin(\Omega t) & \cos(\Omega t) & 0 \\
-\sin(\Omega t) & 0 & 0 & \cos(\Omega t)
\end{pmatrix} \begin{pmatrix}
x_+(t) \\
p_+(t) \\
x_-(t) \\
p_-(t)
\end{pmatrix},$$

after which, due to the conservation of $\bar{p}_L$, the Heisenberg equations read
\[ \cos(\Omega t) \partial_t \tilde{x}_+ (t) - \sin(\Omega t) \partial_t \tilde{p}_- (t) = \sqrt{\frac{2}{T}} \kappa \tilde{p}_L (ct, 0), \]
\[ \sin(\Omega t) \partial_t \tilde{x}_- (t) + \cos(\Omega t) \partial_t \tilde{p}_+ (t) = 0, \]
\[ -\sin(\Omega t) \partial_t \tilde{p}_+ (t) + \cos(\Omega t) \partial_t \tilde{x}_- (t) = 0, \]
\[ \sin(\Omega t) \partial_t \tilde{x}_+ (t) + \cos(\Omega t) \partial_t \tilde{p}_- (t) = 0, \]
\[ \partial_t \tilde{x}_L (\xi, t) = \sqrt{\frac{2}{T}} \kappa \delta (ct - \xi) (\cos(\Omega t) \tilde{p}_+ (t) + \sin(\Omega t) \tilde{x}_- (t)), \]
\[ \partial_t \tilde{p}_L (\xi, t) = 0. \]

They are solved by
\[ \tilde{x}_+ (t) = \tilde{x}_+ (0) + \sqrt{\frac{2}{T}} \kappa \int_0^t d\tau \cos(\Omega \tau) \tilde{p}_L (c\tau, 0), \]
\[ \tilde{p}_+ (t) = \tilde{p}_+ (0), \]
\[ \tilde{x}_- (t) = \tilde{x}_- (0) - \sqrt{\frac{2}{T}} \kappa \int_0^t d\tau \sin(\Omega \tau) \tilde{p}_L (c\tau, 0), \]
\[ \tilde{p}_- (t) = \tilde{p}_- (0), \]
\[ \tilde{x}_L (\xi, t) = \tilde{x}_L (\xi, 0) + \sqrt{\frac{2}{T}} \kappa \left( \cos \left( \frac{\Omega \xi}{c} \right) \tilde{p}_+ (\frac{\xi}{c}) + \sin \left( \frac{\Omega \xi}{c} \right) \tilde{x}_- (\frac{\xi}{c}) \right), \]
\[ \tilde{p}_L (\xi, t) = \tilde{p}_L (\xi, 0). \]

With the approximation \((\Omega T)^{-1} \ll 1\), the input and output operator definitions and the methods from the previous section, we can derive the relations
\[ x_{\text{out}}^+ = x_{\text{out}}^+ + \kappa p_{\text{in}}^+, \]
\[ p_{\text{out}}^+ = p_{\text{out}}^+ , \]
\[ x_{\text{out}}^- = x_{\text{out}}^- - \kappa p_{\text{in}}^-, \]
\[ p_{\text{out}}^- = p_{\text{out}}^- , \]
\[ x_{\text{out}}^c = x_{\text{out}}^c + \kappa p_{\text{in}}^c, \]
\[ p_{\text{out}}^c = p_{\text{out}}^c , \]
\[ x_{\text{out}}^s = x_{\text{out}}^s + \kappa x_{\text{in}}^s, \]
\[ p_{\text{out}}^s = p_{\text{out}}^s . \]

A notable and convenient feature of this configuration is that there is no backaction of light onto itself. Because of the antipodal magnetic fields, \(\tilde{p}_+\) and \(\tilde{p}_-\) are conserved quantities, and the backaction terms in the calculations analogous to (3.11) vanish.

Here the main idea of the protocol can be seen: by measuring the commuting \(x_{\text{out}}^c\) and \(x_{\text{out}}^s\), one can gain information about the atomic variables \(p_+\) and \(x_-\). It is possible to squeeze the non-local atomic modes and create entanglement this way.

By making use of the Gaussian formalism presented in Chapter 2 and rewriting the input-output-relations for the effective modes as a Gaussian operation like in (2.23), the process so far is conveniently described:
\[ R_{\text{out}} = \begin{pmatrix} x_{\text{out}}^+ \\ p_{\text{out}}^+ \\ x_{\text{out}}^- \\ p_{\text{out}}^- \\ x_{\text{s}}^- \\ p_{\text{s}}^- \end{pmatrix} = SR_{\text{in}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \kappa & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{\text{in}}^+ \\ p_{\text{in}}^+ \\ x_{\text{in}}^- \\ p_{\text{in}}^- \\ x_{\text{s}}^- \\ p_{\text{s}}^- \end{pmatrix}. \]

If all states are in the vacuum before the interaction, \( \gamma_{\text{in}} = 1_8 \),

\[ \gamma_{\text{out}} = S\gamma_{\text{in}}S^\dagger = \begin{pmatrix} 1 + \kappa^2 & 0 & 0 & 0 & \kappa & 0 & 0 \\ 0 & 1 & 0 & \kappa & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & 1 + \kappa^2 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 & 1 + \kappa^2 & 0 & 0 \\ \kappa & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \kappa & 0 & 0 & 0 & 1 + \kappa^2 \\ 0 & 0 & 0 & -\kappa & 0 & 0 & 0 \end{pmatrix}. \quad (4.5) \]

Now, we apply the measurement of \( x_{\text{s}}^\text{out} \) and \( x_{\text{c}}^\text{out} \). On the covariance matrix level, the operation can be described with \( \gamma_{\text{M}} = \text{diag}(1/g_{x,c}, g_{x,c}, 1/g_{x,s}, g_{x,s}) \) in the limit \( g_{x,c} \to \infty, g_{x,s} \to \infty \), \[ \gamma_{\text{out}} = S\gamma_{\text{in}}S^\dagger \]

This is a pure two-mode-squeezed state as introduced in Section 2.2.4. The operators \( R = \{ x_+, p_+, x_-, p_- \}^T \) remain in this description, and we see that \( p_+ \) and \( x_- \) are squeezed. We can read off the variances and calculate the EPR-variance \( (2.38) \)

\[ \Delta_{\text{EPR}} = (\Delta x_-)^2 + (\Delta p_+)^2 = \frac{1}{2} \frac{1}{(1 + \kappa^2)} + \frac{1}{2} \frac{1}{(1 + \kappa^2)} = \frac{1}{1 + \kappa^2} < 1. \]

The magnetic field is responsible for the permanent interchange of the atomic variables leading to two-mode-squeezing. Without the field, the final state would still be entangled, but with just one EPR-mode squeezed.

### 4.2 State of the Art

We are interested in two-mode squeezed states between two atomic modes and focus on deterministic entanglement generation like in the original proposal \[30\]. This is
opposed to the probabilistic entanglement generation heralded by detection of single photons, which is used for example in the DLCZ-protocol \[27, 28\] for quantum repeaters \[29\] with atomic ensembles. The DLCZ-protocol relies on parametric gain and beam splitter interaction between light and ensembles and creates Bell states in two collective spins.

Two things have to be emphasized about the deterministic scheme at the heart of this thesis. First, the entanglement between the ensembles is measurement-induced. After the pure QND-interaction of the two atomic ensembles with the coherent light, there exists only entanglement between each ensemble and the perpendicularly polarized light mode, but not between the ensembles themselves, see (4.5). It is the measurement that transforms the correlations between light and atomic modes - the off-diagonal non-zero elements in the covariance matrix - into correlations between the ensembles. Progress has been made in measurement-induced entanglement generation since then, for deterministic as well as for heralded entanglement \[81, 82, 83, 84\].

Encouraged by achievements in other systems \[85, 86\], recent work on the above setup shows a way to generate entanglement between the atomic ensembles without explicitly measuring, purely induced by interaction with the laser beam \[32, 33\]. More specifically, entanglement is created as an interference effect between two indistinguishable processes: Both ensembles interact with the same electromagnetic field modes, and the respective origins of the scattered photons cannot be determined. It is shown, that for an interaction Hamiltonian of the form \[3.6\] a real off-balance between beam-splitter and parametric gain contribution is necessary, and that the above QND-interaction does not meet the requirements for the scheme to work. In the experiment \[33\] the realistic Hamiltonian of Section \[3.2\] was used.

Second, the homogeneous magnetic field that was applied in the early as well as in the recent experimental implementations, is a very important point. With the use of one laser pulse, non-local squeezing was achieved in just one of the two EPR-modes in the original proposal. In order to obtain squeezing also in the other one, the suggestion was to then rotate the variables and repeat the process. As was shown in the previous chapter, the magnetic field accomplishes this by Larmor rotation of the atomic quadratures.

### 4.3 Outline of the Thesis’ Ansatz

The central goal of this thesis is to create in the setup described above as fast as possible as much entanglement as possible. The analysis is done in the general abstract framework of the Gaussian formalism, in which our atomic ensemble setup can be described, see Chapter \[3\]. Inspired by the discussion above, we follow three strategies to meet our goal, and because of the achievements with different quadratic Hamiltonians, we do all our considerations for the general Hamiltonian \[3.6\].

First, we investigate whether measurement results may be used for feedback to generate faster or more entanglement - provided, that measurements are applied. Second,
we examine if and which measurements can help to improve the entanglement generation scheme in case they are not already part of the protocol. And third, it is explored, whether the homogeneous magnetic field is already the optimal option to influence the atomic variables for more or faster entanglement creation - how can supplementary intermediate operations like temporally varying magnetic fields improve the protocols?

In Chapter 5 we start with the first strategy by considering the setup with a measurement operation. We analyze whether the results can be used in order to control the system to faster produce more entanglement. This is what is called feedback. Motivated by recent achievements with a special form of feedback, the so-called linear or displacing feedback, we investigate the help of more general feedback in general Gaussian entanglement generation: general operations, which depend on the measurement results, are applied to the system’s constituents. For our special case of entanglement generation between atomic ensembles, this is shown in Fig. 4.2. In this chapter, we develop a complete characterization of the optimal role of feedback for our setup: Feedback can be used for conditioning the output state with its linear version, but all more general forms cannot improve our protocol any further. Also, we show that the linear feedback can be applied as a one-shot method in the end instead of by using the common continuous form.

Although we have proven that measurement results cannot be used for feedback, measurements as operations may still be of help. Similarly, although we show that additional operations cannot be chosen more optimal, if depending on measurement results, they still can help to increase the output entanglement as operations independent of the measurement results. Encouraged by successes for similar tasks, we...

\[ \text{Figure 4.2: In Chapter 5 we investigate the question, whether supplementary operations like } R \equiv O_\alpha \oplus O_\beta \oplus O_\gamma \text{ can be chosen optimally independently of the measurement results. This is the question of general feedback. } O_\alpha, O_\beta \text{ and } O_\gamma \text{ denote passive operations on the atomic ensembles and the light field, respectively. Together they are treated as one feedback operation } R. \]
After having ruled out the use of their results, we want to determine the help of measurements as entangling operations. Also, we are interested in how to choose the supplementary local passive operations $O_\alpha$, $O_\beta$ and $O_\eta$ optimally to achieve as fast as possible as much entanglement as possible. We deal with both cases of laser polarization, here polarization in $x$-direction is depicted as an example.

In Chapter 6 we therefore turn to strategies two and three, see Fig. 4.3. In particular, we consider just local passive operations for the supplementary ones, as non-local and active operations could generate entanglement on their own. In experiments, they can be realized as waveplates in the light beam or as magnetic fields on the atomic ensembles.

Our numerical investigations show that measurements of proper light quadratures help to create entanglement faster. Also, supplementary operations on the atoms can be used to obtain faster entanglement generation. However, our studies show that the entanglement evolution for an optimal choice of these operations is exactly the same as for usual antipodal homogeneous magnetic fields like in present experiments. We also give an analytic proof that special additional operations on the atoms cannot improve the entanglement evolution at all, if such a constant magnetic field is applied.

Note that in Chapter 6 we do not take into account any noise, whereas the feedback Chapter 5 is also true for Gaussian noise during the interactions.
5 Feedback Assisted Entanglement Generation

This chapter is the first main part of this thesis and deals with the question of feedback in our setup. The first section contains the main statement: a general no-go-result for feedback in Gaussian conditional state entanglement generation. It is shown, that the optimal set of supplementary operations does not depend on the measurement results. However, we leave open the possibility here, that they can help as measurement-result independent operations. Also, with an optimal choice of operations, every state, that is conditioned on a particular set of measurement results shows the same entanglement.

Up to now, we studied feedback protocols for a completely general Gaussian system. In the second section, entanglement in a more specific system is considered, namely in a bipartite system of $1 \times 1$ bosonic modes. Measurements are conducted on correlated auxiliary systems. Clearly, this treatment suits to atomic ensembles and the light field, see Fig. 4.2. We show, that the conditional output states resulting from the optimal choice of additional operations are never less entangled than any unconditional state gained by any additional procedure in the above sense.

But still, there is one task left, which can be completed by feedback: Turning an unconditional into one of the desired conditional states. It is well known that its linear version serves this purpose [38], and the third section is therefore devoted to its study. In a general setup, we show, that all measurements can equivalently be performed in the very end, and that the continuous linear feedback may be replaced by a one-shot linear feedback operation after the measurements.

5.1 No Feedback for Conditional States

In this section, we deal with a Gaussian state with covariance matrix $\gamma$ and displacement vector $d$ undergoing various Gaussian evolution steps, which are indicated by the indices 1, 2, .... Fig. 5.1 illustrates the process in the following proposition.

**Proposition** Consider a system in a Gaussian state, on which arbitrary trace-preserving entangling Gaussian operations $\{G_1, G_2, \ldots\}$ and Gaussian measurements $\{M_1, M_2, \ldots\}$ with results $\{d_{M,1}, d_{M,2}, \ldots\}$ are performed in predetermined order on subsystems. Add a set of intermediate local Gaussian operations $F \equiv \{R_1(d_{M,1}), R_2(d_{M,1}, d_{M,2}), \ldots\}$ to maximize the entanglement in the respective conditional output state.
Then, for all measurement results the optimal choice of the additional operations $F_{\text{opt}}$ for this purpose is the same and therefore independent of the measurement results. For $F_{\text{opt}}$, all conditional output states, conditioned each on a different set of measurement results, are equally entangled.

Note, that the additional operations might still serve the purpose of increasing the output state’s entanglement. But as they do not depend on the measurement results, it is not a feedback scheme. So, feedback is of no help to increase the conditional state’s entanglement in Gaussian entanglement generation.

**Proof**  The proof of this statement is based on three major observations:

i) Entanglement of a Gaussian state is, as a global property, independent of the displacement vector and just a feature of the covariance matrix.

ii) The effect of Gaussian operations on covariance matrix level is independent of the input state’s displacement vector.

iii) Gaussian measurements are deterministic on covariance matrix level.

The first observation was already discussed in Section 2.2.3 and the second follows immediately from equation (2.32). In consequence, all entanglement properties can be studied solely on the level of covariance matrices and the displacements be left out. The third can be understood by identifying the POVM $\{ |\gamma_{M,i}, d_{M,i}\rangle \langle \gamma_{M,i}, d_{M,i}| : d_{M,i} \in \mathbb{R}^n \}$ with measurement $M_i$: The probabilistic application of any of these effects changes the input covariance matrix $\gamma$ in the same way. Just the displacement vector $d$ is changed differently for different applied effects, see (2.33).

The concept of the proof can be understood best by considering Fig. 5.2 which essentially sketches all possible evolutions that may occur in the general protocol of Fig. 5.1 for one given set of supplementary operations $F$. Start at the bottom of the tree with the input covariance matrix $\gamma_1$. First, apply the operation $\mathcal{G}_1$ and the measurement $M_1$. A result $d_{M,1} = x_1$ is obtained probabilistically, where $x_1 \in X = \{ 0,1 \}$.

**Figure 5.1:** A general Gaussian feedback protocol sketched in time. Entangling operations $\mathcal{G}_i$ and measurements $M_i$ are combined and represented as one block, as they operate together deterministically on covariance matrix level, see text. There is the opportunity of an additional potential feedback operation $R_i$ after each measurement.

[Diagram of feedback protocol]
\{A, B, C, \ldots\} \text{ and } X \text{ is the class of measurement results: three different evolution branches can be taken. However, due to iii), the updated covariance matrix is } \gamma_2 \text{ with probability one, no matter which measurement result was obtained.}

Now apply in every branch the respective first element of } F. \text{ The covariance matrix is updated differently in each branch to } \gamma_3(x_1). \text{ Then the next operation } G_2 \text{ and the next measurement } M_2 \text{ is applied in every branch; in Fig. 5.2 for ease of notation this is just drawn for the branch to the right. Different measurement results } d_{M,2} = x_2 \text{ for } x_2 \in X' = \{A', B', C', \ldots\} \text{ can appear, and each branch splits up further. The according operations } R_{2}(x_1, x_2) \text{ are applied and so on.}

In the end, we have for a given } F \text{ one branch for every series of successive measurement results. For all possible } F, \text{ we get one such tree. Now consider all branches of all trees and look for the optimally entangled output state. It is achieved for a specific series of measurement results } x^{opt} = \{d_{M,1} = x_1^{opt}, d_{M,2} = x_2^{opt}, \ldots\} \text{ and an according choice of supplementary operations } R^{opt}(x^{opt}) = \{R_1^{opt}(x_1^{opt}), R_2^{opt}(x_1^{opt}, x_2^{opt}), \ldots\}. \text{ Let the tree of Fig. 5.2 now be the one with the branch leading to the optimally entangled state for } x^{opt} = \{d_{M,1} = C, d_{M,2} = A', \ldots\}. \text{ It is drawn boldly, and the optimal choice of operations is here } R^{opt}(x^{opt}) = \{R_1^{opt}(C), R_2^{opt}(C, A'), \ldots\}.

We define the set } F^{opt} \text{ as the one applying } R^{opt}(x^{opt}) \text{ for every series of measurement results, i. e. for all branches. Suppose now, in an actual run of the protocol using } F^{opt}, \text{ we get the series of results } \tilde{x}. \text{ In Fig. 5.2 let } \{\tilde{x} = A, B', \ldots\}. \text{ Because}

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{tree.png}
\caption{A tree for a set of supplementary operations } F, \text{ which is later assumed to be the one with the optimal branch, see text. Open arrows always indicate further branches of the tree that are not sketched here.}
\end{figure}
the Gaussian operations of $R^{opt}$ operate deterministically on covariance matrix level, ii), we still achieve the covariance matrix of the optimal branch in every step. This is shown as the double-lined path in the figure. Therefore, for the optimal choice $F^{opt}$ we end up with the optimally entangled state for all series of measurement results. □

This section ends with three closing remarks. First, note, that it is not immediately clear, whether all $R_i$ can be optimized separately in every step or just all at once, when one really tries to find the optimal choice. However, the proof works without relying on that.

Second, in experiments, it is also always interesting to know, how well a protocol for entanglement generation performs on average. The proposition gives a convenient answer to this question: No matter, which measurement results appear with whatever probability, by applying the global optimal choice of additional operations the output state will in any case be optimally entangled.

As a last comment, it is emphasized, that in this optimal case the state’s displacement vector can be calculated with (2.32) as soon as the measurement results are given. It is random, but known in the end and can be used for displacing feedback as explained in the last section.

5.2 The Optimally Entangled is a Conditional State

So far, it has been proven, that feedback cannot help to increase the entanglement of the conditional output state any further. On the other hand, unconditional states appear in case of averaging over the measurement results. This averaging over every measurement result can be performed at the very end of the whole evolution, because every integral or sum over a measurement result may be interchanged with all other independent evolution terms. For a common entanglement measure for mixed and pure states, the question arises, whether an unconditional state can be better entangled by the use of any $F$ than an optimal conditional one.

**Proposition**  Assuming the system above to be bipartite with just two modes, a conditional state created with the globally optimal additional operations $F^{opt}$ is never less entangled than any unconditional state created with any (feedback) procedure $F$.

**Proof**  According to [57], any bipartite two-mode state is, up to local Gaussian unitaries, equivalent to a two-mode-squeezed state (TMSS, (2.18)) characterized by one parameter. A pure TMSS with entanglement parameter $r$ can be transformed deterministically to any other TMSS with $r'$ by means of Gaussian local operations and classical communication (GLOCC), iff $r \geq r'$.

Consider now a conditional output state $|\Psi_{\vec{m}}^{F}\rangle$ for measurement results $\vec{m} \equiv (m_1, ..., m_n)$ and a procedure $F$ in the sense of the previous section. Because the system is now
assumed to be bipartite, and because the state is less entangled than the state resulting from $F_{\text{gopt}}$, $|\Psi_{\text{gopt}}\rangle$, it can be created deterministically with GLOCC from the latter. Note, that the unitaries from the transformation to TMSSs can be included in the GLOCC conveniently.

With the help of a dice, for every probability $p_{m}$ one can therefore create the according state $|\Psi_{m}\rangle$, yielding altogether the unconditional state $\rho^{\text{uc}} \equiv \sum_{m} p_{m} |\Psi_{m}^{F}\rangle \langle \Psi_{m}^{F}|$ again by means of GLOCC. As every entanglement measure $E$ at least has to be monotone, see (2.34), therefore the inequality $E(|\Psi_{\text{gopt}}\rangle \langle \Psi_{\text{gopt}}|) \geq E(\rho^{\text{uc}})$ holds. \hfill $\square$

5.3 Conditioning in the End

In this section, some considerations about the implementation of monitoring, i. e. continual measurements, and continuous linear feedback are made. The section is formulated in terms of a system of interest, $T$, interacting with a large number of systems $\{S_{1}, ..., S_{n}\}$ one after the other. $\{S_{1}, ..., S_{n}\}$ are in a next step subject to measurements. To include noise, we consider the interactions as unitary Gaussian operations on both the respective system and an environment $E$, see (2.27). Yet, the unitaries are described in the formalism of general Gaussian operations, see Section 2.3.3, and the displacements are fully taken into account. Alternatively, a treatment in terms of Gaussian channels as introduced on covariance matrix level in 2.3.2 would be possible.

While the calculations hold independently of the implementation and therefore for general Gaussian setups, they apply immediately to our setup, where $T$ describes the atomic ensembles and $\{S_{1}, ..., S_{n}\}$ the discretized light modes.

**Proposition** Independent of whether the measurements are executed continuously or after the whole interaction, the system under control, $T$, experiences the same change. Similarly, the continuous linear feedback may be replaced by a one-shot linear feedback operation in the end.

**Proof** Just considering two systems $S_{1}$ and $S_{2}$, the state $\rho$ under consideration is defined on the Hilbert space $\mathcal{H} = \mathcal{H}_{S_{1}} \otimes \mathcal{H}_{S_{2}} \otimes \mathcal{H}_{T} \otimes \mathcal{H}_{E}$, and the result can be extended easily to $n$ systems by induction. Unitaries on systems $X$, $Y$, ... with corresponding covariance matrix $\Gamma$ and displacement vector $D$ are in this chapter denoted by $U^{\Gamma,D}_{X,Y,...}$. For example, $U^{\Gamma_{1},D_{1}}_{S_{1},T,E}$ is the noisy interaction between $S_{1}$ and $T$, and $U^{\Gamma_{\text{id}},D(m_{1}/2)}_{T}$ the displacing feedback on $T$ depending on the measurement result $m_{1}/2$. The measurement operator on system $S_{1/2}$, e. g. a projection, is denoted by $M_{S_{1/2}}(m_{1}/2)$, and $S_{1}$, $S_{2}$ and $E$ are traced out in the end, as just the state of $T$ is of interest. This treatment of measurements includes the homodyne detections of (2.33).

Denoting the probabilities for the measurement outcomes
\[ p(m_1) \equiv \text{tr}_{S_1, T,E} \{ U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} M_{S_1}^\dagger(m_1) \}, \]
\[ p(m_2) \equiv \text{tr}_{S_2, T,E} \{ U_{S_2, T,E}^{\Gamma_2, D_2} U_T^{\Gamma_2, D_2} M_{S_2}(m_2) \frac{M_{S_2}(m_2)}{p(m_2)} U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} M_{S_2}(m_2) \dagger \}, \]

and using the fact, that operators, which act non-trivially just on different subspaces, commute, one can calculate

\[
\rho_T = \text{tr}_{S_1, S_2, E} \{ U_T^{\Gamma_1, D_1} M_{S_2}(m_2) \frac{M_{S_2}(m_2)}{p(m_2)} U_{S_2, T,E}^{\Gamma_1, D_1} U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} 
\]
\[
= \text{tr}_{S_1, S_2, E} \{ U_T^{\Gamma_1, D_1} M_{S_2}(m_2) \frac{M_{S_2}(m_2)}{p(m_2)} U_{S_2, T,E}^{\Gamma_1, D_1} U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} 
\]
\[
= \text{tr}_{S_1, S_2, E} \{ U_T^{\Gamma_1, D_1} \frac{M_{S_2}(m_2)}{p(m_2)} U_{S_2, T,E}^{\Gamma_1, D_1} M_{S_2}(m_2) \frac{M_{S_2}(m_2)}{p(m_2)} U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} 
\]
\[
= \text{tr}_{S_1, S_2, E} \{ U_T^{\Gamma_1, D_1} \frac{M_{S_2}(m_2)}{p(m_2)} U_{S_2, T,E}^{\Gamma_1, D_1} \frac{M_{S_2}(m_2)}{p(m_2)} U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} 
\]
\[
= \text{tr}_{S_1, S_2, E} \{ U_T^{\Gamma_1, D_1} \frac{M_{S_2}(m_2) M_{S_2}(m_2)}{p(m_2)} U_{S_2, T,E}^{\Gamma_1, D_1} \frac{M_{S_2}(m_2) M_{S_2}(m_2)}{p(m_2)} U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} 
\]
\[
= \text{tr}_{S_1, S_2, E} \{ U_T^{\Gamma_1, D_1} \frac{M_{S_2}(m_2) M_{S_2}(m_2)}{p(m_2)} U_{S_2, T,E}^{\Gamma_1, D_1} \frac{M_{S_2}(m_2) M_{S_2}(m_2)}{p(m_2)} U_{S_1, T,E}^{\Gamma_1, D_1} \rho(\ldots) \} 
\]

Here, \((\ldots)^\dagger\) stands for the Hermitian conjugate of all operations to the left of \(\rho\).

It remains to show, that interchanging \(U_T^{\Gamma_1, D_1}\) and \(U_{S_2, T,E}^{\Gamma_1, D_2}\) at the second equality sign preserves for the former the form of a pure displacement. Also, we have to determine its value \(\tilde{D}(m_1)\). With the notation

\[
\Gamma_2 = \begin{pmatrix}
\Gamma_{2,A} & \Gamma_{2,C} \\
\Gamma_{2,C} & \Gamma_{2,B}
\end{pmatrix}, \quad D(m_1) = \begin{pmatrix} D(m_1)_A \\ D(m_1)_B \end{pmatrix}, \quad D_2 = \begin{pmatrix} D_{2,A} \\ D_{2,B} \end{pmatrix},
\]

we take a closer look at the operations’ action

\[
U_T^{\Gamma_1, D_1} : (\gamma, \delta) \mapsto (\gamma', \delta'(D(m_1), \delta)) = (\gamma, D(m_1)_A + D(m_1)_B + \delta),
\]
\[
U_{S_2, T,E}^{\Gamma_2, D_2} : (\gamma', \delta') \mapsto (\gamma'', (\Gamma_2, \gamma), \delta''(\Gamma_2, D_2, \gamma, \delta')).
\]

The calculation
\[(\gamma'', d'') = (\gamma''(\Gamma_2, \gamma), D_{2,A} + \hat{\Gamma}_{2,C} \frac{1}{\Gamma_{2,B} + \gamma} (D_{2,B} + d'))
\]
\[= (\gamma''(\Gamma_2, \gamma), D_{2,A} + \hat{\Gamma}_{2,C} \frac{1}{\Gamma_{2,B} + \gamma} (D_{2,B} + D(m_1)_A + 
\quad + D(m_1)_B) +
\quad + \hat{\Gamma}_{2,C} \frac{1}{\Gamma_{2,B} + \gamma} (D(m_1)_A + D(m_1)_B))
\]
\[= (\gamma''(\Gamma_2, \gamma), d''(\Gamma_2, D_2, \gamma, d) +
\quad + \hat{\Gamma}_{2,C} \frac{1}{\Gamma_{2,B} + \gamma} (D(m_1)_A + D(m_1)_B))
\]

then shows, that with the definition

\[\tilde{D}(m_1) \equiv (\hat{\Gamma}_{2,C} \frac{1}{\Gamma_{2,B} + \gamma} (D(m_1)_A + D(m_1)_B), 0)
\]

one gets

\[U_{S_2, T, E}^{\Gamma_2, D_2} \circ U_T^{\Gamma_2, d, D(m_1)} = U_T^{\Gamma_2, \tilde{D}(m_1)} \circ U_{S_2, T, E}^{\Gamma_2, D_2}.
\]

So, the new feedback operation can be calculated with the description of the interaction, as soon as the measurement results are known.

Finally, it is shown, that the probabilities for the measurement results do not change during the above interchange, and that therefore it is justified to always use \(p(m_1)\) and \(p(m_2)\) instead of new probabilities \(\tilde{p}(m_1)\) and \(\tilde{p}(m_2)\) in the calculations above.

Local unitaries on subspaces which are traced out do not change the reduced state. They just change the basis, in which the basis-independent trace is taken:

\[\text{tr}_B\{(\mathbb{1}_A \otimes U_B)\rho_{AB}(\mathbb{1}_A \otimes U_B)^\dagger\} = \sum_i \langle i|U_B \rho_{AB}(U_B^\dagger|i) \rangle_B
\]
\[= \text{tr}_B\{\rho_{AB}\}.
\]

Therefore, together with the commutation ideas from above,
\[ \tilde{p}(m_1) \equiv \text{tr}_{S_1} \left\{ M^{S_1}(m_1) \text{tr}_{S_2,T,E} \left\{ U_{T_2,D_2}^{S_2,T,E} U_{T_1,D_1}^{S_1,T,E} \rho(\ldots) \right\} M^{S_1}(m_1) \right\} \]
\[ = \text{tr}_{S_1} \left\{ M^{S_1}(m_1) \text{tr}_{S_2,T,E} \left\{ U_{T_1,D_1}^{S_1,T,E} \rho(\ldots) \right\} M^{S_1}(m_1) \right\} \]
\[ = p(m_1), \]

\[ \tilde{p}(m_2) \equiv \text{tr}_{S_2} \left\{ M^{S_2}(m_2) \text{tr}_{S_1,T,E} \left\{ \frac{M^{S_1}(m_1)}{p(m_1)} U_{T_2,D_2}^{S_2,T,E} U_{T_1,D_1}^{S_1,T,E} \rho(\ldots) \right\} M^{S_2}(m_2) \right\} \]
\[ = \text{tr}_{S_2} \left\{ M^{S_2}(m_2) \text{tr}_{S_1,T,E} \left\{ U_{T_1,D_1}^{S_1,T,E} \frac{M^{S_1}(m_1)}{p(m_1)} U_{T_2,D_2}^{S_2,T,E} \rho(\ldots) \right\} M^{S_2}(m_2) \right\} \]
\[ = \text{tr}_{S_2} \left\{ M^{S_2}(m_2) \text{tr}_{S_1,T,E} \left\{ U_{T_1,D_1}^{S_1,T,E} \frac{M^{S_1}(m_1)}{p(m_1)} U_{T_2,D_2}^{S_2,T,E} \rho(\ldots) \right\} M^{S_2}(m_2) \right\} \]
\[ = p(m_2). \]

The results might be helpful in the design of future experiments, in which the question of when to measure and when to do linear feedback arises. While the replacement of continuous linear feedback by one feedback operation in the end can be done easily in experiments, removing all measurements to the very end seems rather impractical. In the implementation of atomic ensembles with laser light, for example, the light modes would have to be stored until the end of the interaction process. Anyway, the measurement considerations are important in giving the experimenter theoretical reasoning to freely choose the distance of the measurement devices from the atomic ensembles, as this corresponds to an interchange of some measurements with interaction operations.
6 Local Operation Assisted Entanglement Generation

This chapter is the second main part of the thesis, and it is also completely written in the Gaussian formalism. After having found in the previous chapter, that intermediate supplementary operations cannot be chosen more optimal, if depending on measurement results, here we determine an actual optimal set independent of them. We investigate also whether and which homodyne light measurements can help to reach the goal of faster producing more entanglement, see Fig. 4.3.

In general there are many possibilities, in which supplementary operations can be applied to the atomic ensembles’ modes and to the light mode. In 6.1, we motivate our choice of allowed operations and derive a differential equation for the system’s evolution in this case. We treat a very general quadratic interaction Hamiltonian that includes (3.6) and are guided by a similar equation in the special case of QND-interaction [88].

In 6.2 we present numerical results for the optimization of the system’s entanglement evolution, that are obtained on the basis of this differential equation. The results are interpreted in 6.3. We find that certain measurements help to accelerate the entanglement generation, if they are not necessary for the protocol anyway like for QND-interaction. In addition, our specific study of supplementary light rotations shows, which quadrature has to be measured in the optimal case.

For operations on the ensembles we gain strong evidence that the optimal procedure corresponds to the application of antipodal static homogeneous magnetic fields on the ensembles. In 6.4 we prove analytically, that special additional atomic operations can not increase the entanglement, if these magnetic fields are already applied. This is a first step towards a better understanding of the underlying physics in our optimization studies.

Note that we work in an ideal setup in the whole chapter. Further studies which include noise in our system could be worth to be investigated, as then general additional operations of our form could prove more useful.

6.1 The Setup and its Description

We consider two atomic ensembles $A_1$ and $A_2$ that interact with a light mode $L$. The interaction is identical at both ensembles and given by the general quadratic Hamiltonian $H_{int}$. Section 6.1.1 introduces a standard form of $H_{int}$. Up to local
passive operations on $A_1$, $A_2$ and $L$, it gives rise to an evolution that is described by a special $S$-matrix in phase space representation, see Section 2.3.1.

In 6.1.2 we then apply this standard form and get

$$\gamma_A(t + \delta t) = M_x R^T S_2 S_1 R \gamma_{AL}(t) (\ldots)^T$$  \hspace{1cm} (6.1)

as the basic update formula of our setup in one time step. $\gamma_A$ is the $4 \times 4$ covariance matrix of $A_1$ and $A_2$ and contains all information about the entanglement between them. In every time step, the ensembles interact by $S_1$ and $S_2$ with a new vacuum light mode $L$ with covariance matrix $\mathbb{I}_2$, i.e. they act on $\gamma_{AL} = \gamma_A \oplus \mathbb{I}_2$. In the end, a homodyne measurement of the light’s $x$-quadrature $M_x$ reduces the $6 \times 6$ matrix to $\gamma_A$ again. If no measurement is applied, the light modes are simply traced out.

$R = O_{A_1} \oplus O_{A_2} \oplus O_L$ are local passive operations on atomic ensembles $A_1$, $A_2$ and on the current light mode $L$. We have their application as magnetic fields at the ensembles and as waveplates in the light field in mind. All $O_i$ ($i = A_1, A_2, L$) are assumed to be generated by the Hamiltonians

$$H_{\text{loc},i} \propto (x_i^2 + p_i^2)$$  \hspace{1cm} (6.2)

cf. (3.9), to be applicable instantaneously and to be tunable at will. With the formalism of Section 2.3.1, they can be easily shown to correspond to rotations in phase space by angles $r = \alpha, \beta, \eta$,

$$O_i = \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix}.$$

The local operations that bring $H_{\text{int}}$ into the standard form are included in these rotations for easier calculations.

Our specific choice of the additional operations corresponds to a potential basis change in all three subsystems $A_1$, $A_2$ and $L$. On the one hand, $O_L$ allows to separate the basis of interaction from the measurement basis. Optimizing the light rotations therefore characterizes the quadrature of the light mode that is best measured. On the other hand, the atomic operations $O_{A_1}$ and $O_{A_2}$ allow for dynamic control of which general atomic quadratures best interact with which quadratures of the light field.

Finally, in Section 6.1.3, we calculate an explicit differential equation for $\gamma_A$ from (6.1). Together with the goal to maximize the entanglement $E$ at an end time $T$, we end up with what is called an optimal control problem in the literature:

$$\dot{\gamma}_A(t) = D + E \gamma_A(t) + \gamma_A(t) E^T - \gamma_A(t) F \gamma_A(t)$$

$$E(\gamma_A(T)) \nearrow.$$  \hspace{1cm} (6.3)

$D, E$ and $F$ depend on the angles $\alpha, \beta$ and $\eta$, that are time-dependent functions. The optimization is carried out with respect to them. For this class of problems,
analytical results are only known for linear cost functions and quadratic evolution equations. Because our problem involves solving an eigenvalue problem for calculating our cost function, the entanglement, we have to draw on numerics in Section 6.2.

Note that to be precise the treatment in (6.1) is only exact for alternating between light scattering and additional rotations. That means, that the laser has to be switched off, when the rotations are applied. A proper treatment of the evolution, if all interactions happen at the same time, can be obtained by using the Trotter formula [1]. To first order, that also yields the above equation.

Finally, we remark, that the techniques developed in this chapter can be used for optimization within other sets of operations. It would for example be very interesting to investigate the simulation of different Hamiltonians at both ensembles. This could be achieved with an additional light operation between the application of $S_1$ and $S_2$ above. It seems promising to first have a two-mode-squeezer Hamiltonian at $A_1$ for high entanglement between $A_1$ and $L$, and then a beam splitter between $A_2$ and $L$ to transfer it to entanglement between $A_1$ and $A_2$.

### 6.1.1 Standard Form of the Interaction

We first consider the most general form of a quadratic interaction Hamiltonian $H_{int}$ between two bosonic modes. Since we want to apply it to the case of an atomic ensemble and a light mode, we denote their canonical variables $(x_A, p_A)$ and $(x_L, p_L)$. We leave out locally applicable terms of the form (6.2), as they can be included in the supplementary local operations,

$$H_{int} = ax_Ax_L + bp_Ap_L + cp_Ax_L + dx_Ap_L = (x_A \quad p_A) K \begin{pmatrix} x_L \\ p_L \end{pmatrix}, \quad K \equiv \begin{pmatrix} a & d \\ c & b \end{pmatrix}.$$

According to the singular value decomposition (SVD), there exist orthogonal matrices $\tilde{O}_1, \tilde{O}_2 \in O(2, \mathbb{R})$ and $\tilde{D} \equiv \begin{pmatrix} \tilde{s}_1 & 0 \\ 0 & \tilde{s}_2 \end{pmatrix}$ with $\tilde{s}_1 \geq \tilde{s}_2 \geq 0$, such that $K = \tilde{O}_1 \tilde{D} \tilde{O}_2^T$. Because for every $\tilde{O}_i \in O(2, \mathbb{R})$ it holds, that $\det(\tilde{O}_i) = 1$ or $\det(\tilde{O}_i) = -1$, either directly $\tilde{O}_i \in SO(2, \mathbb{R})$ or $\tilde{O}_i = \tilde{Q}_i \sigma_3$, where $\tilde{Q}_i \in SO(2, \mathbb{R})$ is a special orthogoanl matrix and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If $\det(K) > 0$, $\det(\tilde{O}_1)$ and $\det(\tilde{O}_2^T)$ have the same sign due to $\det(\tilde{D}) > 0$. By doing nothing in the positive and replacing $\tilde{O}_i$ by $\tilde{Q}_i$ in the negative case for both $i = 1, 2$, we can choose $\tilde{O}_1, \tilde{O}_2 \in SO(2, \mathbb{R})$ in the SVD above because $\sigma_3 \tilde{D} \sigma_3^T = \tilde{D}$.

If $\det(K) < 0$, on the other hand, just one of $\tilde{O}_1$ and $\tilde{O}_2$ can have negative determinant. We replace the one with the negative determinant by its respective $\tilde{Q}_i$ and keep the other.

Since $\sigma_3 \tilde{D} = \tilde{D} \sigma_3^T = \begin{pmatrix} \tilde{s}_1 & 0 \\ 0 & -\tilde{s}_2 \end{pmatrix}$, we can therefore choose in all cases
\[ K = \tilde{O}_1 \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \tilde{O}_2^T, \]

where now \( \tilde{O}_1, \tilde{O}_2 \in SO(2, \mathbb{R}) \) for \( s_1 \equiv \tilde{s}_1 \) and \( s_2 \equiv \text{sign} \{ \text{det} K \} \tilde{s}_2 \). This is the restricted singular value decomposition. As \( \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( \tilde{O} \in SO(2, \mathbb{R}) \) commute, it can be calculated, that with the definition \( X \equiv \begin{pmatrix} 0 & s_2 \\ -s_1 & 0 \end{pmatrix} \),

\[ \sigma_2 K = \tilde{O}_1 \sigma_2 \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \tilde{O}_2^T = \tilde{O}_1 X \tilde{O}_2^T, \]

\[ \sigma_2 K^T = \tilde{O}_2 \sigma_2 \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \tilde{O}_1^T = \tilde{O}_2 X \tilde{O}_1^T. \]

With the notation introduced in Section 2.3.1, we have the \( 4 \times 4 \) matrices \( \tilde{M} = \begin{pmatrix} K \end{pmatrix} \) and

\[ \tilde{G} = \sigma \tilde{M} + \sigma \tilde{M}^T = \begin{pmatrix} \sigma_2 K^T & \sigma_2 K \end{pmatrix} = \begin{pmatrix} \tilde{O}_1 \oplus \tilde{O}_2 \end{pmatrix} \begin{pmatrix} X & X \end{pmatrix} \begin{pmatrix} \tilde{O}_1 \oplus \tilde{O}_2 \end{pmatrix}^T. \]

for the operator order \( (x_A, p_A, x_L, p_L) \). So, in phase space representation the time evolution governed by a general quadratic Hamiltonian can always be described by

\[ S = \exp[\delta t \tilde{G}] = (\tilde{O}_1 \oplus \tilde{O}_2) \exp \left[ \delta t \begin{pmatrix} X & X \end{pmatrix} \right] (\tilde{O}_1 \oplus \tilde{O}_2)^T. \]

The notation of (products of) \( 2 \times 2 \) matrices at entries of another matrix is in this chapter always to be understood blockwise. Empty spaces stand for blocks of according size filled with 0s, and \( (\tilde{O}_1 \oplus \tilde{O}_2) = \begin{pmatrix} \tilde{O}_1 \\ \tilde{O}_2 \end{pmatrix} \) then. For example,

\[ \begin{pmatrix} X & X \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & s_2 \\ 0 & 0 & -s_1 & 0 \\ 0 & s_2 & 0 & 0 \\ -s_1 & 0 & 0 & 0 \end{pmatrix}. \]

### 6.1.2 Including Supplementary Operations

Now we turn to the actual problem of three bosonic modes: \( A_1, A_2 \) and \( L \). We choose the operator order \( (x_{A_1}, p_{A_1}, x_{A_2}, p_{A_2}, x_L, p_L) \) and apply the results of the previous section blockwise. The discrete formulation of light modes as introduced in \( 3.1.2 \) is used, and so we have to deal with the additional factor of \( 1/\sqrt{\delta t} \). By absorbing the global \( 1/\sqrt{\delta t} \) in \( X \), one ends up with \( 6 \times 6 \) matrices \( S_{1/2} \equiv \exp(\sqrt{\delta t} G_{1/2}) \), where
\[ G_1 = (\tilde{O}_1 \oplus 1 \oplus \tilde{O}_2) \begin{pmatrix} X \\ X \end{pmatrix} (\tilde{O}_1 \oplus 1 \oplus \tilde{O}_2)^T, \]

\[ G_2 = (1 \oplus \tilde{O}_1 \oplus \tilde{O}_2) \begin{pmatrix} X \\ X \end{pmatrix} (1 \oplus \tilde{O}_1 \oplus \tilde{O}_2)^T. \]

Now we include supplementary physical operations \( \bar{R} = (\bar{O}_A \oplus \bar{O}_A \oplus \bar{O}_L) \) before and \( \bar{R}^T \) after the interaction step of duration \( \delta t \). We calculate

\[ \bar{R}^T \tilde{S}_2 \tilde{S}_1 \bar{R} = \bar{R}^T (1 \oplus \tilde{O}_1 \oplus \tilde{O}_2) \exp \left[ \sqrt{\delta t} \begin{pmatrix} X \\ X \end{pmatrix} \right] (\tilde{O}_1 \oplus \tilde{O}_1^T \oplus 1) \]

\[ \times \exp \left[ \sqrt{\delta t} \begin{pmatrix} X \\ X \end{pmatrix} \right] (\tilde{O}_1^T \oplus 1 \oplus \tilde{O}_2^T) \bar{R} \]

\[ = \bar{R}^T (\tilde{O}_1 \oplus \tilde{O}_1 \oplus \tilde{O}_2) S_2 S_1 (\tilde{O}_1^T \oplus \tilde{O}_1^T \oplus \tilde{O}_2^T) \bar{R} \]

\[ = \bar{R}^T S_2 S_1 R, \]

where we defined \( R \equiv (O_1 \oplus O_2 \oplus O_3) \equiv (\tilde{O}_1^T \oplus \tilde{O}_1^T \oplus \tilde{O}_2^T) \bar{R} \) and

\[ S_{1/2} \equiv \exp[\sqrt{\delta t}G_{1/2}], \quad G_1 \equiv \begin{pmatrix} X \\ X \end{pmatrix}, \quad G_2 \equiv \begin{pmatrix} X \\ X \end{pmatrix}. \]

Note that the physical rotations are the \( \bar{R} \), and therefore all optimal results \( R_{opt} \) calculated in the following have to be converted back by means of the rotations that bring \( H_{int} \) into standard form. Application of a homodyne measurement of the light’s \( x \)-quadrature \( M_x \) in the end then yields (6.1).

### 6.1.3 Optimal Control Problem

To obtain a differential equation for \( \gamma_{A} \) from (6.1), it is enough to keep just the terms to order \( \sqrt{\delta t} \) here in a Taylor expansion, as can be justified in the end.

\[ R^T S_2 S_1 R \gamma_{AL}(t) R^T S_1^T S_2^T R \]

\[ = \gamma_{AL}(t) + \sqrt{\delta t}[R^T(G_1 + G_2)R_{\gamma_{AL}}(t) + h.c.] + \delta t[R^T(G_1^2 + G_2^2) + 2G_1 G_2]R_{\gamma_{AL}}(t) + h.c.] \]

\[ + \delta t R^T(G_1 + G_2)R_{\gamma_{AL}}(t) R^T(G_1^T + G_2^T) R + O(\delta t^{3/2}) \]

The terms can be calculated blockwise:
\[R^T(G_2 + G_1)R\gamma_{AL}(t) = \begin{pmatrix} \tilde{F}_2 \gamma_A(t) \\ \tilde{F}_1 \end{pmatrix},\]

\[R^T(G_2 + G_1)R\gamma_{AL}(t)R^T(G_1^T + G_2^T)R = \begin{pmatrix} \tilde{D} & \tilde{B}_1 \gamma_A(t) \tilde{B}_1^T \\ \tilde{E} \gamma_A(t) & \tilde{B}_2 \end{pmatrix},\]

\[R^T\left(\frac{G_2^2}{2} + \frac{G_1^2}{2} + G_2G_1\right)R\gamma_{AL}(t) = \begin{pmatrix} \tilde{E} \gamma_A(t) \\ \tilde{B}_2 \end{pmatrix}\]

for the auxiliary matrices

\[\tilde{B}_1 \equiv (O_1^T X_1 O_1 \quad O_3^T X_1 O_2),\]
\[\tilde{F}_1 \equiv \begin{pmatrix} O_1^T X_1 O_3 \\ O_2^T X_1 O_3 \end{pmatrix},\]
\[\tilde{D} \equiv \begin{pmatrix} O_1^T X_2^T O_1 & O_1^T X_3^T O_2 \\ O_2^T X_2^T O_1 & O_2^T X_3^T O_2 \end{pmatrix},\]
\[\tilde{E} \equiv \begin{pmatrix} O_1^T X_1^2 O_1 & O_3^T X_1^2 O_2 \\ O_2^T X_1^2 O_1 & O_2^T X_2^2 O_2 \end{pmatrix} \]

of format \(2 \times 2\) in the first, \(4 \times 2\) in the second and \(4 \times 4\) in the third line. Before the measurement, we therefore have to lowest order in \(\delta t\) the \(6 \times 6\) matrix

\[R^T S_2 S_1 R\gamma_{AL}(t) R^T S_1^T S_2^T R = \begin{pmatrix} \gamma_A + \delta t [\tilde{D} + \tilde{E} \gamma_A(t) + \gamma_A(t) \tilde{E}^T] + \sqrt{\delta t} \left[ \tilde{F}_1 + \gamma_A(t) \tilde{F}_1^T \right] & \sqrt{\delta t} \left[ \tilde{F}_1 + \gamma_A(t) \tilde{F}_1^T \right] \\ \sqrt{\delta t} \left[ \tilde{F}_1 + \gamma_A(t) \tilde{F}_1^T \right] & \mathbb{I}_2 + \delta t B \end{pmatrix}\]

for \(B = \tilde{B}_1 \gamma_A(t) \tilde{B}_1^T + \tilde{B}_2 + \tilde{B}_2^T\). The upper left block is here of size \(4 \times 4\), and the lower right of \(2 \times 2\). The upper left entry of \(B\) is denoted by \(b_{11}\). The Moore-Penrose inverse is, see Section 2.3.3

\[(\pi (\mathbb{I} + \delta t B) \pi)^{MP} = \begin{pmatrix} \frac{1}{1 + \delta t b_{11}} & 0 \\ 0 & 0 \end{pmatrix} = M_x - \delta t \begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix} + O(\delta t^2)\]

for \(M_x \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\). Applying the homodyne measurement update formula (2.33) finally yields to lowest order in \(\delta t\)

\[\gamma_A(t + \delta t) = \gamma_A(t) + \delta t [\tilde{D} + \tilde{E} \gamma_A(t) + \gamma_A(t) \tilde{E}^T] - \sqrt{\delta t} \left[ \tilde{F}_1 + \gamma_A(t) \tilde{F}_1^T \right] \left[ M_x - \delta t \begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix} \right] \sqrt{\delta t} \left[ \tilde{F}_2 \gamma_A(t) + \tilde{F}_2^T \right] = \gamma_A(t) + \delta t \{D + E \gamma_A(t) + \gamma_A(t) E^T - \gamma_A(t) F \gamma_A(t)\}\]
for the $4 \times 4$ matrices

\[ D \equiv \tilde{D} - \tilde{F}_1 M_x \tilde{F}_1^T \\
= (O_1^T \oplus O_2^T) \left( \begin{array}{cc}
X(1 - O_3 M_x O_3^T)X^T & X(1 - O_3 M_x O_3^T)X^T \\
X(1 - O_3 M_x O_3^T)X^T & X(1 - O_3 M_x O_3^T)X^T
\end{array} \right) (O_1 \oplus O_2),
\]

\[ E \equiv \tilde{E} - \tilde{F}_1 M_x \tilde{F}_2 \\
= (O_1^T \oplus O_2^T) \left( \begin{array}{cc}
X(\frac{1}{2}1 - O_3 M_x O_3^T)X & -XO_3 M_x O_3^T X \\
X(\frac{1}{2}1 - O_3 M_x O_3^T)X & X(\frac{1}{2}1 - O_3 M_x O_3^T)X
\end{array} \right) (O_1 \oplus O_2),
\]

\[ F \equiv \tilde{F}_2^T M_x \tilde{F}_2 \\
= (O_1^T \oplus O_2^T) \left( \begin{array}{cc}
X^T O_3 M_x O_3^T X & X^T O_3 M_x O_2^T X \\
X^T O_3 M_x O_2^T X & X^T O_3 M_x O_2^T X
\end{array} \right) (O_1 \oplus O_2).
\]

From these equations we can explicitly see, that the light operations $O_L$ are nothing but a change of basis of the measurement operation. Also it can be understood here, how the atomic operations $O_1$ and $O_2$ act as basis transformations: The evolution equation for $\tilde{\gamma}_A(t) = (O_1 \oplus O_2)\gamma_A(t)(O_1 \oplus O_2)^T$ has the same form as (6.3), where $D, E$ and $F$ are as in (6.4) for $O_1 = O_2 = I_2$. The additional atomic operations therefore determine the basis, in which the evolution step takes place.

The evolution without explicit measurements is contained in this formula by setting $M_x = 0$. This is the differential equation for the optimal control problem (6.3).

### 6.2 Numerical Studies

We choose time steps $\delta t = 0.001$ and calculate the covariance matrix evolution with the Euler method for 20 time steps, that are labeled by $n$. We choose the vacuum $\gamma_A(t = 0) = 1_4$ as the initial condition and the logarithmic negativity from Section 2.4.2 as the measure of entanglement. Three cases are compared in the following.

First, we calculate the entanglement, when no rotations are applied. Second, we optimize the logarithmic negativity of the general covariance matrix at $T = 20 \times 0.001$ as a function of $3 \times 20 = 60$ angle variables. And third, we optimize the logarithmic negativity stepwise, i. e. we optimize after every single of the 20 timesteps over 3 angles and calculate with that the covariance matrix of the next step. It is far from obvious, that local and global optimization procedure yield the same results, as in a very similar problem the local method was not able to find the global maximum [34].

The results for evolution with and without measurement are presented for two exemplary Hamiltonians, the $QND$- and the realistic Hamiltonian, see Section 3.2. The physical interaction Hamiltonian can be embedded in the abstract formulation of this chapter by setting $\hbar/\sqrt{\kappa} = 1$, as this global constant just leads to a rescaling of time. Note, that the factor $c^{-1/2}$ comes from the discretization of light modes. Then, $s_1 = Z$ and $s_2 = 1/Z$. 
6.2.1 QND-Hamiltonian

We consider the QND-Hamiltonian for laser light polarized in $x$-direction, $H_{QND} = p_{APL}$, i.e. $s_1 = 0, s_2 = 1$. The final state’s logarithmic negativity is shown for all cases in the following table:

<table>
<thead>
<tr>
<th></th>
<th>no rot.</th>
<th>opt. rot.</th>
<th>stepw. opt. rot.</th>
</tr>
</thead>
<tbody>
<tr>
<td>without measurement</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>with measurement</td>
<td>0.0196481..</td>
<td>0.0198389..</td>
<td>0.0198415..</td>
</tr>
</tbody>
</table>

An optimal angle configuration of the local has the same structure as one of the global optimization method. We show such a typical set of optimal angles for the case of applied measurements in Fig. 6.1. In the case without measurement, we cannot generate any entanglement even with the help of rotations and do not show any optimal angle plot hence.

6.2.2 Realistic Hamiltonian

Now the realistic Hamiltonian $H_{int} = 2.5x_Ax_L + 0.4p_{APL}$ is considered. Here, we add calculations of the logarithmic negativity for measurement of $p_L$ instead of $x_L$. This can be simulated by simply replacing $M_x$ with $M_p = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in the above formulae due to an analogous derivation.

<table>
<thead>
<tr>
<th></th>
<th>no rot.</th>
<th>opt. rot.</th>
<th>stepw. opt. rot.</th>
</tr>
</thead>
<tbody>
<tr>
<td>without measurement</td>
<td>0.0167604..</td>
<td>0.0167900..</td>
<td>0.0167900..</td>
</tr>
<tr>
<td>with $x_L$-measurement</td>
<td>0.0167615..</td>
<td>0.0999322..</td>
<td>0.0999323..</td>
</tr>
<tr>
<td>with $p_L$-measurement</td>
<td>0.0944677..</td>
<td>0.0999322..</td>
<td>0.0999323..</td>
</tr>
</tbody>
</table>

A typical optimal angle configuration for local and global optimization has again a similar structure as before in the case of applied $x_L$-measurements, see Fig. 6.1. As can be seen from (6.4), in the case of no applied measurements light rotations drop out already in the evolution equation of the covariance matrix. As they therefore cannot influence the output state’s entanglement, here just the optimal atomic angle configuration is of interest, see also Fig. 6.1. An optimal configuration for $p_L$-measurements has the same form as in the QND case with $x_L$-measurement and is not plotted separately.

6.2.3 Longer Simulations

A first interpretation of the results is, that the global optimum can, within the computational accuracy, usually be found by local optimization. Because the local optimization procedure scales much better in the number of evolution steps $n$, this enables much longer simulations.
Figure 6.1: Numerical solutions for optimal configuration of angles. The solid lines with dots always belong to the rotation angle at $A_1$, $\alpha$, and the dashed lines with squares to the rotation at $A_2$, $\beta$. (a) and (b) show the atomic and light rotations, respectively, in the case of QND-interaction $H_{\text{int}} = p_A p_L$ with measurement of $x_L$. Simulation results for the realistic Hamiltonian $H_{\text{int}} = 2.5 x_A x_L + 0.4 p_A p_L$ are shown in (c)-(e). (c) and (d) denote optimal atomic and light rotations, respectively, for the case of measurement of $x_L$, and (e) the case without any explicit measurement.
Figure 6.2: (a) and (b) show the numerically calculated optimal angle configuration and the optimal entanglement evolution for a simulation of \( n = 2500 \) steps. In (c) and (d) the same simulation is extended to \( n = 25000 \) steps. The simulation is done for the realistic Hamiltonian with \((optimal)\) \( \mu \)-measurements, see text.
Also, we gain numerical evidence for restricting the class of the initial values of the angles, that lead to the optimum. For the atomic angles, we find, that \( \alpha(n) + \beta(n) \) is a constant in the optimal case. Further studies show, that we can still reach the optimal entanglement, if we set this constant to 0, i.e. if we choose \( \alpha(n) = -\beta(n) \).

For the light rotations, we calculate

\[
O_3 M_x O_3^T = \begin{pmatrix} \cos^2(\eta) & -\cos(\eta) \sin(\eta) \\ -\cos(\eta) \sin(\eta) & \sin^2(\eta) \end{pmatrix}
\]

and see, that rotations by \( \eta = 0 \) and \( \eta = \pi \) lead to the same effect. We conclude, that the optimum is in the QND case obtained by leaving out the light rotations at all.

For the realistic Hamiltonian, when measuring \( x_L \), we obtain the same phenomenon: Light rotations of \( \eta = \pi/2 \) and of \( \pi = 3\pi/2 \) yield the same effect. Then \( O_3 M_x O_3^T = M_p \), which means that measurement of \( p_L \) is the overall optimal light operation. This is confirmed by studies with \( p_L \)-measurements like in the previous section. They show, that depending on which quadrature is measured, very different amounts of entanglement are created.

All together, we conclude that we can simulate the optimal configuration by just optimizing locally over one atomic angle. This is done for the realistic Hamiltonian in Fig. 6.2 in the case of (optimal) \( p_L \)-measurement.

### 6.3 Interpretation

#### 6.3.1 Understanding Light Operations

We conduct further numerical studies of optimal light rotations for different interaction Hamiltonians. The result is, that always either \( \eta_{opt} = k\pi, k \in \mathbb{Z} \) or \( \eta_{opt} = \pi/2 + k\pi, k \in \mathbb{Z} \). The former result appears, if \( s_2 > s_1 \), and the latter for \( s_2 < s_1 \). As optimal light rotations only choose the quadrature to be best measured with homodyning, (6.5), the optimal case is either a measurement of \( x_L \) or of \( p_L \). Note that in 6.1.1 these two quadratures were distinguished.

A physical understanding can be gained in terms of two-mode-squeezing. The Hamiltonian determines, which non-local modes are squeezed, and which are anti-squeezed. From the exemplary input-output-equations (4.1, 4.2) for the QND-case it can be seen that light modes with a certain polarization couple to certain non-local modes, and therefore it is also immediately decided, which quadrature to measure in the optimal case. By always choosing the right quadrature, we can therefore omit any supplementary light rotations such as waveplates.

#### 6.3.2 Understanding Atomic Operations

As noted before, an optimal angle configuration is given by \( \alpha(n) = -\beta(n) \) in every iteration step \( n \) for all interaction Hamiltonians. More insight is gained by looking
at longer evolution times, Fig. 6.2. In an optimal configuration $\alpha(n + 1) - \alpha(n) = k\pi/2, k \in \mathbb{Z}$ for almost all $n$. Sometimes, there are other jumps, but the spacing stays almost everywhere the same. We zoom in for a typical interval, Fig. 6.3, and see, that here $\alpha(n + 1) - \alpha(n) \neq 0$ for all $n$ - the angle always changes by a finite multiple of $\pi/2$.

Motivated by these considerations, we set $\alpha_{opt}(n) = n\pi/2$ (and $\beta_{opt}(n) = -n\pi/2$). We compute for this choice that for all $n$ the logarithmic negativity is exactly the same as the one that was found to be the numerical optimum. We conclude, that the irregularities in Fig. 6.2 can be affiliated to numerical errors and that we have found an optimal configuration within our set of allowed operations.

To obtain a physical understanding of this solution, we forget about the interpretation of the rotations as giving rise to a basis transformation now. We consider again (6.1): If we drop out the light operations and always measure the optimal quadrature, $R = O_1 \oplus O_2 \oplus 1_2$ acts non-trivially only on the atomic subspaces. It commutes with $M_{x/p}$, as the latter is just a projection operation in the light subspace. Then we can write

\[ ...M_{x/p}R((n + 1)\pi/2)^T S_2 S_1 R((n + 1)\pi/2)M_{x/p}R(n\pi/2)^T S_2 S_1 R(n\pi/2)... \]

\[ = ...M_{x/p}R((n + 1)\pi/2)^T S_2 S_1 R((n + 1)\pi/2)R(-n\pi/2)M_{x/p}S_2 S_1 R(n\pi/2)... \]

\[ = ...M_{x/p}R_{eff}(\pi/2) S_2 S_1 R_{eff}(\pi/2)M_{x/p}S_2 S_1 R_{eff}(\pi/2) S_2 S_1... \]
Our optimal solution is nothing but the application of $\pi/2$-rotations after every pulse. Such a rotation interchanges, up to a minus sign, the canonical variables of the ensembles. It leads to non-local two-mode squeezing in the sense of \[2.2.4\]. In more detailed numerical studies we find for the realistic Hamiltonian squeezing of $x_+$ and $p_-$, while the QND-Hamiltonian squeezes $x_-$ and $p_+$. If to the contrary no rotations are applied at all, in the QND-case a state with non-local one-mode squeezing is prepared. It is worse entangled according to \[2.38\].

An optimal strategy that keeps interchanging the atomic variables reminds a lot of the effect of the static homogeneous magnetic field already applied in experiments \[31\]. The fields are chosen antipodal to each other, which is essentially reflected in our numerical results $\alpha_{opt}(n) = -\beta_{opt}(n)$.

We therefore conjecture, that the "usual" application of magnetic fields is already an optimal strategy. To challenge this idea, we contrast our simulations of optimal entanglement evolution to evolution equations for applied magnetic fields as derived in \[32, \!33\]. These equations are given for the EPR-variance \[2.38\] instead of for the logarithmic negativity. Like for our simulations, we discretize the light modes in them and set the global prefactors to 1 to make them comparable. Then, for the two cases of $p_L$-measurements and without any explicit measurements, they are given by

$$\Delta_{EPR,p_L}(t) = \frac{1}{e^{-t} + Z^2(1 - e^{-t})}, \quad \Delta_{EPR,wo}(t) = e^{-t} + \frac{1}{Z^2}(1 - e^t).$$

(6.6)
For our numerical studies, we have to take a representative specific Hamiltonian - we choose the realistic one. Because in this case $x_+$ and $p_-$ are squeezed, we can easily calculate the EPR-variance from the covariance matrix in each step,

$$\Delta_{EPR}(t) = \frac{1}{2} \left[ (\Delta(x_{A1} + x_{A2}))^2(t) + (\Delta(p_{A1} - p_{A2}))^2(t) \right]$$

$$= \frac{1}{4} \text{tr}\{\gamma(t)\} + \frac{1}{2} [\gamma_{A,13}(t) + \gamma_{A,24}(t)].$$

The result is plotted in Fig. 6.4. We gain strong numerical evidence, that magnetic fields have the same effect as our set of optimal atomic rotations, no matter whether light measurements are applied or not. In 6.4 we make a first step in the direction of analytically proving the magnetic fields’ optimality.

### 6.3.3 Measurements vs. No Measurements

Although their results cannot be used for feedback, measurements accelerate the entanglement generation as operations. This is the result of many numerical studies, and as an example plot we refer again to Fig. 6.4. The same observation can be made in 6.6, when antipodal static magnetic fields are applied.

Already in Fig. 6.2 it could be seen, that the entanglement is bounded from above for the realistic Hamiltonian. This behavior has already been investigated in 32 for the case of applied antipodal static magnetic fields. It was shown for a general Hamiltonian of the form (3.6), that for the asymptotic limit

$$\Delta_{EPR} \to (|\mu| - |\nu|)^2 \quad (t \to \infty).$$

It is exactly this limit that we reach with our simulation as well. We can add now, that neither measurements nor supplementary operations of the above form can increase it.

### 6.4 Magnetic field: Additional operations do not help

As an optimal solution of supplementary operations and measurements leads to exactly the same entanglement evolution as for applied antipodal magnetic fields, we consider a similar setup as in Fig. 4.1 again. We now assume a general interaction Hamiltonian 3.6 and $y$-polarization of the laser beam, and we measure without loss of generality $p_L$. Still, the ensembles are spin-polarized in $x$-direction, and the magnetic fields, which give rise to Larmor frequencies $\mp \Omega$, are oriented in $\mp x$-direction at ensemble $A1/A2$. Note, that Larmor rotations of opposite sign can equivalently be realized by antipodal spin-alignment and parallel magnetic fields. This would mean preparation of the second atomic ensemble in $|g\rangle = |F = 4, m_F = -4\rangle$, but we choose the first option here.
In Appendix B we motivate an effective Hamiltonian

\[ H_{\text{int}} = \frac{\hbar \kappa}{\sqrt{c T \delta t}} [Z(x_{Ac}x_{Lc} + x_{As}x_{Ls}) + \frac{1}{Z} (p_{Ac}p_{Lc} + p_{As}p_{Ls})] \equiv R^TMR \]

for the effective interaction between both atomic ensembles and discretized light modes. Because magnetic fields are applied, similar to 3.3.3 (co)sine modes for light are introduced. The concept is extended to (co)sine modes of the atomic ensembles, which are similar to the known non-local modes of \[ x_{A,c} = x_+, p_{A,c} = p_+, x_{A,s} = -p_- \text{ and } p_{A,s} = x_- \].

This section shows, that supplementary atomic rotations of opposite sign cannot further improve the output state's entanglement. It is the conclusion of a straightforward calculation, which is enabled by this effective Hamiltonian and conducted in the basis given by the operators introduced above, \[ R(c/s) = (x_{Ac}, p_{Ac}, x_{As}, p_{As}, x_{Lc}, p_{Lc}, x_{Ls}, p_{Ls}) \]. Objects in this basis are indexed \((c/s)\), whereas the "regular" local basis has indices \((1/2)\). For the short hand notation \( \tau = \frac{\hbar \kappa}{\sqrt{c T \delta t}} \), we have

\[ S = \exp\{\delta t G\} = \exp\{\delta t (\sigma M + \sigma M^T)\} \]

Rotations on atomic ensembles \( R(1/2) = O_1(\alpha) \oplus O_2(-\alpha) \oplus I_4 \) can be translated into the \((c/s)\)-basis with the transformation \( U \) from \((B.1)\):

\[ R(c/s) = (U \oplus I_4) R(1/2)(U \oplus I_4)^{-1} = \begin{pmatrix} \cos(\alpha) & 0 & \sin(\alpha) & 0 \\ 0 & \cos(\alpha) & 0 & \sin(\alpha) \\ -\sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & -\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \oplus I_4. \]

If we start now with a state with covariance matrix \( \gamma_{AL(c/s)} = \text{diag}(\frac{1}{2}, d, \frac{1}{2}, d, 1, 1, 1, 1) \), we just have to use the measurement operator’s covariance matrix \( \text{diag}\{g_p, 1/g_p, g_p, 1/g_p\} \) \((2.33)\) in \((6.1)\):

\[ \gamma_{A,(c/s)} = M_x(R^T_{c/s}) S R(c/s) \gamma_{AL,(c/s)}(...)^T \]

\[ = \text{diag}\{\frac{1}{d'}, d', \frac{1}{d'}, d'\} \]
such a step cannot depend on it. From the initial vacuum state with $d = 1$ we therefore prepare a two-mode-squeezed state, which cannot get better entangled by any supplementary atomic rotations of opposite sign.
7 Application: Quantum State Teleportation between Atomic Ensembles

In [30] it is described briefly how to teleport the state $|\Psi^{in}\rangle$ described by $x^{in}_{\Psi}, p^{in}_{\Psi}$ of an atomic ensemble $\Psi$ next to an ensemble $A$ of Alice to Bob’s ensemble $B$ far away. The basis are the general three teleportation steps introduced in Section 2.5. It is suggested to create the entanglement between $A$ and $B$ with the far off-resonant QND-interaction similar to the treatment in [41]. As one cannot directly measure the atomic ensembles’ states, the following measurement step may then also be performed on the basis of the same interaction with light beams.

There are several ways to implement these steps by varying how many light beams are used, i.e., the other way round, how many are reused for different purposes. Two vicarious schemes for these thoughts are considered in the following in a noiseless setting. Scheme I is an approach using different light beams for every task. Scheme II reuses the entangling light beam as part of the EPR-measurement step, which needs a laser pulse anyway.

The performance of the protocols is assessed with the average fidelity between $|\Psi^{in}\rangle$ and the final state of $B$ for a class of coherent states with a Gaussian distribution. Also, it is then compared to the quantum benchmark for quantum state transfer [42, 43]. We find, that Scheme I performs better than Scheme II, and is even able to beat the quantum benchmark, if the to be teleported state is chosen from a relatively small class.

While two-mode-squeezing was achieved by an artificial perpetual exchange of atomic variables in [30], with the knowledge of the discussions so far the schemes here make extensive use of magnetic fields on the ensembles. In the treatise of both schemes, just the relevant input-output-relations for important points in each step are given; these points are enumerated by Greek letters. In contrast to former parts of this thesis, measurement and linear feedback are described as a whole with a formalism explained in Section C.

7.1 Scheme I

In the straight forward scheme as depicted in Fig. 7.1 entanglement generation between $A$ and $B$ and EPR-measurement on Alice’s side are completely separated.
Figure 7.1: Straight forward teleportation Scheme I with the possibility to feed back with gains $g_1$ and $g_2$ on $x_B$, and with $g_3$ and $g_4$ on $p_B$. As before, the grey squares are the atomic ensembles, and the black arrows indicate their spin polarizations. Light grey arrows are the applied homogeneous magnetic fields. The relevant variables at the important points indicated by greek letters are given in the text. Light beams are labeled $i$, $ii$ and $iii$, and the beam splitter is sketched by a vertical bar.

Two measurements are conducted, and one can feed back two measurement results on each of $B$'s quadratures to optimize the performance of the protocol.

**Entanglement generation between A and B** The first step was demonstrated in detail in Section 4.1. To be precise in transferring, $A$ and $B$ have to be interchanged, but this is allowed, as here just the ideal case is considered. So, before the measurement, from equations (2.19) [4.1] [4.2], in the rotating frame we get

\[
\begin{align*}
\alpha & ) \quad x_{\text{out}}^{B,i} = x_{\text{in}}^{B,i} + \frac{\kappa}{\sqrt{2}} p_{c,i}^{\text{in}}, \\
p_{\text{out}}^{B,i} = p_{\text{in}}^{B,i} + \frac{\kappa}{\sqrt{2}} p_{s,i}^{\text{in}}, \\
x_{\text{out}}^{A,i} = x_{\text{in}}^{A,i} + \frac{\kappa}{\sqrt{2}} p_{c,i}^{\text{in}}, \\
p_{\text{out}}^{A,i} = p_{\text{in}}^{A,i} - \frac{\kappa}{\sqrt{2}} p_{s,i}^{\text{in}},
\end{align*}
\]
\[ x_{s,i}^{\text{out}} = x_{s,i}^{\text{in}} + \frac{\kappa}{\sqrt{2}} x_{A,i}^{\text{in}} + \frac{\kappa}{\sqrt{2}} x_{B,i}^{\text{in}}, \]  
\[ x_{c,i}^{\text{out}} = x_{c,i}^{\text{in}} + \frac{\kappa}{\sqrt{2}} p_{A,i}^{\text{in}} + \frac{\kappa}{\sqrt{2}} p_{B,i}^{\text{in}}. \]  
(7.5)  
(7.6)

Here, because the interaction is mediated by the light beam \( i \), all variables in this step carry the according index.

**EPR-Measurement** The second step is conducted with the use of light beams \( \text{ii} \) and \( \text{iii} \) incident on \( A \) and \( \Psi \), respectively. The atomic ensembles’ states are mapped to the light fields, and by superposing the latter on a 50:50 beam splitter, the resulting light modes \( x_{s/c,\pm}^{\text{out}} \equiv (x_{s/c,\pm}^{\text{out}} + x_{s/c,\pm}^{\text{out}})/\sqrt{2} \) carry information about atomic EPR-modes \( (x_{A,\text{ii}}^{\text{in}} \pm x_{\Psi,\text{iii}}^{\text{in}})/\sqrt{2} \) and \( (p_{A,\text{ii}}^{\text{in}} \pm p_{\Psi,\text{iii}}^{\text{in}})/\sqrt{2} \). So, measurement of the atomic variables is achieved by measuring appropriate light modes.

In the derivation of the relevant input-output-relations right before the actual measurements, one has to take care of the magnetic fields’ antipodal orientation. This is straightforward to see and feed it back with gains \( g_1, g_2 \) to \( x_{B}^{\text{out}} \). Analogous, we proceed with \( x_{c,i}^{\text{out}}, x_{c,-}^{\text{out}} \) and \( p_{B}^{\text{out}} \) by using gains \( g_3 \) and

\[ x_{s,\text{ii}}^{\text{out}} = x_{s,\text{ii}}^{\text{in}} + \frac{\kappa}{\sqrt{2}} x_{A,\text{ii}}^{\text{in}} + \frac{\kappa^2}{2} p_{c,\text{ii}}^{\text{in}} + \frac{\kappa^2}{4} p_{c,\text{ii}}^{\text{in}} + \frac{\kappa^2}{4} p_{c,\text{ii}}^{\text{in}}, \]  
\[ x_{c,\text{ii}}^{\text{out}} = x_{c,\text{ii}}^{\text{in}} + \frac{\kappa}{\sqrt{2}} p_{A,\text{ii}}^{\text{in}} - \frac{\kappa^2}{2} p_{s,\text{ii}}^{\text{in}} - \frac{\kappa^2}{4} p_{s,\text{ii}}^{\text{in}} - \frac{\kappa^2}{4} p_{s,\text{ii}}^{\text{in}}, \]  
\[ x_{s,\text{iii}}^{\text{out}} = x_{s,\text{iii}}^{\text{in}} - \frac{\kappa}{\sqrt{2}} x_{\Psi,\text{iii}}^{\text{in}} - \frac{\kappa^2}{2} p_{c,\text{iii}}^{\text{in}} - \frac{\kappa^2}{4} p_{c,\text{iii}}^{\text{in}} - \frac{\kappa^2}{4} p_{c,\text{iii}}^{\text{in}}, \]  
\[ x_{c,\text{iii}}^{\text{out}} = x_{c,\text{iii}}^{\text{in}} + \frac{\kappa}{\sqrt{2}} p_{\Psi,\text{iii}}^{\text{in}} + \frac{\kappa^2}{2} p_{s,\text{iii}}^{\text{in}} + \frac{\kappa^2}{4} p_{s,\text{iii}}^{\text{in}} + \frac{\kappa^2}{4} p_{s,\text{iii}}^{\text{in}}, \]  
\[ x_{s,\text{\pm}}^{\text{out}} = x_{s,\text{\pm}}^{\text{in}} + \frac{\kappa}{2} (x_{A,\text{\pm}}^{\text{in}} \mp x_{\Psi,\text{\pm}}^{\text{in}}) + \frac{\kappa^2}{2} p_{c,\text{\pm}}^{\text{in}} + \frac{\kappa^2}{4} p_{c,\text{\pm}}^{\text{in}} + \frac{\kappa^2}{4} p_{c,\text{\pm}}^{\text{in}}, \]  
\[ x_{c,\text{\pm}}^{\text{out}} = x_{c,\text{\pm}}^{\text{in}} + \frac{\kappa}{2} (p_{A,\text{\pm}}^{\text{in}} \pm p_{\Psi,\text{\pm}}^{\text{in}}) - \frac{\kappa^2}{2} p_{s,\text{\pm}}^{\text{in}} - \frac{\kappa^2}{4} p_{s,\text{\pm}}^{\text{in}} - \frac{\kappa^2}{4} p_{s,\text{\pm}}^{\text{in}}. \]  
(7.7)  
(7.8)  
(7.9)  
(7.10)  
(7.11)  
(7.12)

**Classical Communication** Now, we measure \( x_{s,\text{\pm}}, x_{s,\text{\pm}}^{\text{out}} \) and feed it back with gains \( g_1, g_2 \) to \( x_{B}^{\text{out}} \). Analogous, we proceed with \( x_{c,\text{\pm}}, x_{c,\text{\pm}}^{\text{out}} \) and \( p_{B}^{\text{out}} \) by using gains \( g_3 \) and
The variables of $B$’s final state after the whole procedure are denoted $(x/p)_{B}^{fin}$ and easily calculated by

$$x_B^{fin} = x_B^{out} + g_1 x_{s,i}^{out} + g_2 x_{s,+}^{out}$$

$$= (1 - \frac{g_1 \kappa}{\sqrt{2}}) x_B^{in} + \kappa (1 + \frac{g_2 \kappa}{2}) x_{c,i}^{in} + \frac{\kappa}{\sqrt{2}} (g_1 + \frac{g_2 \kappa}{2}) x_{A,i}^{in} + g_2 x_{s,+}^{in}$$

$$+ \frac{g_2 \kappa^2}{4} p_{c,-}^{in} - \frac{g_2 \kappa}{2} x_{\Psi,iii}^{in} + \frac{g_2 \kappa^2}{4 \sqrt{3}} p_{c,1,-}^{in},$$

(7.13)

$$p_B^{fin} = p_B^{out} + g_3 x_{c,i}^{out} + g_4 x_{c,-}^{out}$$

$$= (1 + \frac{g_3 \kappa}{\sqrt{2}}) p_B^{in} + \kappa (1 - \frac{g_4 \kappa}{2}) p_{c,i}^{in} + \frac{\kappa}{\sqrt{2}} (g_3 + \frac{g_4 \kappa}{2}) p_{A,i}^{in} + g_4 x_{c,-}^{in}$$

$$- \frac{g_4 \kappa^2}{4} p_{s,+}^{in} - \frac{g_3 \kappa}{2} p_{\Psi,iii}^{in} - \frac{g_4 \kappa^2}{4 \sqrt{3}} p_{s,1,+}^{in}.$$  

(7.14)

Assessment of Performance To determine how well the teleportation scheme works, we need the mean values and variances of $B$’s final state. Assuming that all input operators except for those of the to be teleported mode of $\Psi$ are in the vacuum state with mean $\langle ... \rangle = 0$ and variance $V(... \rangle = 1/2$, we can calculate the input means and variances of all involved effective modes to also be 0 and $1/2$, respectively. For the (co)sine-type modes, since we assume uncorrelated light modes, in the regime $(\Omega T)^{-1} \ll 1$, e. g.

$$\langle x_{s,i}^{in} \rangle = \langle \sqrt{\frac{2}{T}} \int_0^T d\tau \sin(\Omega \tau) \bar{x}_{L,i}(c\tau,0) \rangle = \sqrt{\frac{2}{T}} \int_0^T d\tau \sin(\Omega \tau) \langle \bar{x}_{L,i}(c\tau,0) \rangle = 0,$$

$$V(x_{s,i}^{in}) = \langle (x_{s,i}^{in})^2 \rangle - \langle x_{s,i}^{in} \rangle^2 = \langle \left( \sqrt{\frac{2}{T}} \int_0^T d\tau \sin(\Omega \tau) \bar{x}_{L,i}(c\tau,0) \right)^2 \rangle$$

$$= \frac{2}{T} \int_0^T d\tau' \int_0^T d\tau \sin(\Omega \tau) \sin(\Omega \tau') \langle \bar{x}_{L,i}(c\tau,0) \bar{x}_{L,i}(c\tau',0) \rangle$$

$$= \frac{2}{T} \int_0^T d\tau' \int_0^T d\tau \sin(\Omega \tau) \sin(\Omega \tau') \frac{1}{2} \delta(\tau - \tau')$$

$$= \frac{1}{2} \frac{2}{T} \int_0^T d\tau \sin^2(\Omega \tau)$$

$$\approx \frac{1}{2},$$

and because there are also no correlations between different light beams, e. g.

$$V(x_{s,+}^{in}) = \frac{1}{2} V(x_{s,i}^{in}) + \frac{1}{2} V(x_{s,ii}^{in}) + \langle x_{s,iii}^{in} x_{s,iii}^{in} \rangle - \langle x_{s,i}^{in} \rangle \langle x_{s,iii}^{in} \rangle = \frac{1}{2}$$
for the ±-type modes. Also, we assume |Ψ\text{in}\rangle to be coherent, i.e., \( V(x_\Psi) = V(p_\Psi) = 1/2 \). Because all involved modes are independent, from (7.13, 7.14) then the final means and variances of B's mode are readily calculated as

\[
\langle x_{B,\text{fin}} \rangle = -\frac{g_2 \kappa}{2} \langle x_{\Psi,\text{ini}} \rangle,
\]
\[
\langle p_{B,\text{fin}} \rangle = -\frac{g_4 \kappa}{2} \langle p_{\Psi,\text{ini}} \rangle,
\]
\[
V(x_{B,\text{fin}}) = g_1^2 \left( \frac{\kappa^2}{2} + \frac{1}{2} \right) + g_2^2 \left( \frac{5 \kappa^4}{48} + \frac{\kappa^2}{4} + \frac{1}{2} \right) - g_1 \frac{\kappa}{\sqrt{2}} + g_2 \frac{\kappa^3}{2 \sqrt{2}} + \frac{\kappa^2}{4} + \frac{1}{2},
\]
\[
V(p_{B,\text{fin}}) = g_3^2 \left( \frac{\kappa^2}{2} + \frac{1}{2} \right) + g_4^2 \left( \frac{5 \kappa^4}{48} + \frac{\kappa^2}{4} + \frac{1}{2} \right) + g_3 \frac{\kappa}{\sqrt{2}} - g_4 \frac{\kappa^3}{4} + g_3 g_4 \frac{\kappa^2}{2 \sqrt{2}} + \frac{\kappa^2}{4} + \frac{1}{2}.
\]

For a given coherent input state |Ψ\text{in}\rangle, we use the fidelity \( F_{\text{coh}}(\langle x_\Psi \rangle, \langle p_\Psi \rangle) \) of Section 2.5.1 between the final state of B and the desired one as a figure of merit, that compares the states' closeness in the underlying Hilbert space. In (2.39), we have the freedom to change the sign of the final state’s displacement to maximize the fidelity, as we can correct it in case by post-processing of the displacement.

\[
F_{\text{coh}}(\langle x_\Psi \rangle, \langle p_\Psi \rangle) = \frac{2}{\sqrt{1 + 2V(x_B)^2}} \frac{1}{\sqrt{1 + 2V(p_B)^2}} e^{-\langle x_\Psi \rangle^2 (1 + 2V(x_B)^2)^{-1} - \langle p_\Psi \rangle^2 (1 + 2V(p_B)^2)^{-1}}.
\]

Actually, we don’t know the state |Ψ\text{in}\rangle to be teleported, but want the scheme to perform optimally on average over a broad class of inputs. So we draw the input amplitudes according to a Gaussian distribution with mean zero and standard deviation \( n \)

\[
p(\langle x_\Psi \rangle, \langle p_\Psi \rangle) = \frac{1}{\pi n} e^{-\frac{(\langle x_\Psi \rangle)^2 + (\langle p_\Psi \rangle)^2}{2n}}
\]

and calculate the average fidelity

\[
\bar{F}(\kappa, n, g_1, g_2, g_3, g_4) = \int d\langle x_\Psi \rangle \int d\langle p_\Psi \rangle p(\langle x_\Psi \rangle, \langle p_\Psi \rangle) F_{\text{coh}}(\langle x_\Psi \rangle, \langle p_\Psi \rangle)
\]
\[
= \frac{2}{\pi n} \frac{1}{\sqrt{1 + 2V(x_B)^2}} \frac{1}{\sqrt{1 + 2V(p_B)^2}} \int d\langle x_\Psi \rangle e^{-\langle x_\Psi \rangle^2 (1 + 2V(x_B)^2)^{-1} - \langle p_\Psi \rangle^2 (1 + 2V(p_B)^2)^{-1}}
\]
\[
\int d\langle p_\Psi \rangle e^{-\langle p_\Psi \rangle^2 (1 + 2V(p_B)^2)^{-1}}
\]
Therefore, here just two measurement results can be fed back on $B$ with gains $g_1$ and $g_2$. Again, black arrows denote the polarization of the spins, and grey arrows the magnetic fields.

\[
\frac{1}{n} \left[ \left( \frac{1}{2n} \left( 1 + 2V(x_{B}^{\text{fin}}) \right) + \left( 1 + \frac{g_2 \kappa}{2} \right)^2 \right) \left( \frac{1}{2n} \left( 1 + 2V(p_{B}^{\text{fin}}) \right) + \left( 1 - \frac{g_4 \kappa}{2} \right)^2 \right) \right]^{-1/2}
= \frac{2}{\sqrt{1 + 2V(x_{B}^{\text{fin}}) + 2n(1 + \frac{g_2 \kappa}{2})^2} \sqrt{1 + 2V(p_{B}^{\text{fin}}) + 2n(1 - \frac{g_4 \kappa}{2})^2}}.
\]

### 7.2 Scheme II

Now, we consider the setup as shown in Fig. 7.2 where the entangling light beam $i$ is also used as part of the measurement step.

**Entanglement generation between $A$ and $B$** The entangling step is exactly the same, and the input-output-relations at $\alpha$ and $\beta$ can be exactly copied from (7.1-7.6).

**EPR-Measurement** The input-output-relations after the interaction of the new light beam $iii$ with ensemble $\Psi$ are given by $\delta$ from Scheme I, (7.9-7.10). Just right before the actual measurement of the light operators at $\epsilon$, the relations now differ, while the idea remains the same: $(x/p)^{\text{out}}_{A,i}$ and $(x/p)^{\text{in}}_{\Psi}$ were mapped to the respective light modes $x_{s,\pm}^{\text{out}}$ and $x_{c,\pm}^{\text{out}}$. Here, the $\pm$-operators are defined between light beam $i$ and...
iii in complete analogy to the definitions in Scheme I, e. g. \( x_{s,\pm}^o = (x_{s,\pm}^o + x_{s,\pm}^i)/\sqrt{2} \). Contrary to Scheme I, now the measurement beam of \( A \) also carries information about \((x/p)^B_{in,i;i}\):

\[
\begin{align*}
\epsilon) \quad x_{s,\pm}^o &= \frac{1}{\sqrt{2}} \left( x_{s,i}^{in} + \frac{\kappa}{\sqrt{2}} x_{A,i}^{in} + x_{B,i}^{in} \pm \left( x_{s,\pm}^{in} - x_{B,i}^{in} \right) \right), \\
\quad x_{c,\pm}^o &= \frac{1}{\sqrt{2}} \left( x_{c,i}^{in} + \frac{\kappa}{\sqrt{2}} p_{A,i}^{in} + p_{B,i}^{in} \pm \left( x_{c,\pm}^{in} + p_{B,i}^{in} \right) \right).
\end{align*}
\]

**Classical Communication** Now, we measure \( x_{s,-}^o \) and feed it back with gain \( g_1 \) to \( x_B^o \). Analogous, we proceed with \( x_{c,+}^o \) and \( p_B^o \) by using gain \( g_2 \). The variables of \( B \)'s final state after the whole procedure are again denoted \((x/p)^B_{fin}\) and given by

\[
\begin{align*}
x_B^{fin} &= x_B^o + g_1 x_{s,-}^o \\
&= (1 - \frac{g_1}{2}) x_{B,i}^{in} + \frac{\kappa g_1}{2} x_{A,i}^{in} + \frac{\kappa}{\sqrt{2}} x_{s,\pm}^{in} + \left( x_{B,i}^{in} - \frac{g_1}{2} x_{s,\pm}^{in} \right) \\
&+ \frac{g_1 \kappa^2}{4\sqrt{2}} p_{B,i}^{in} + \frac{g_1 \kappa^2}{4\sqrt{6}} p_{c,i}^{in}. \\
p_B^{fin} &= p_B^o + g_2 x_{c,+}^o \\
&= (1 + \frac{g_2}{2}) p_{B,i}^{in} + \frac{\kappa g_2}{2} p_{A,i}^{in} + \frac{\kappa}{\sqrt{2}} p_{s,\pm}^{in} + \left( p_{B,i}^{in} + \frac{g_2}{2} p_{s,\pm}^{in} \right) \\
&+ \frac{g_2 \kappa^2}{4\sqrt{2}} p_{c,i}^{in} + \frac{g_2 \kappa^2}{4\sqrt{6}} p_{c,i}^{in}. \\
\end{align*}
\]

**Assessment of Performance** Along the lines of the presentation and under exactly the same conditions as in Scheme I, one obtains for \( B \)'s final means and variances

\[
\begin{align*}
\langle x_B^{fin} \rangle &= \frac{g_1 \kappa}{2} \langle x_{s,\pm}^{fin} \rangle, \\
\langle p_B^{fin} \rangle &= \frac{g_2 \kappa}{2} \langle px_{s,\pm}^{fin} \rangle, \\
V(x_B^{fin}) &= g_1^2 \left( \frac{\kappa^4}{4} + \frac{3\kappa^2}{8} + \frac{1}{2} \right) - g_1 \frac{\kappa}{2} + \frac{1}{2} + \frac{\kappa^2}{4}, \\
V(p_B^{fin}) &= g_2^2 \left( \frac{\kappa^4}{4} + \frac{3\kappa^2}{8} + \frac{1}{2} \right) + g_2 \frac{\kappa}{2} + \frac{1}{2} + \frac{\kappa^2}{4}.
\end{align*}
\]

The fidelity for just one coherent state \( |\Psi_{fin}\rangle \) is given by

\[
F_{coh}(\langle x_{\Psi}^{fin} \rangle, \langle p_{\Psi}^{fin} \rangle) = \frac{2}{\sqrt{1 + 2V(x_B^{fin})^2(1 + 2V(p_B^{fin})^2)}} e^{-\frac{(x_{\Psi}^{fin})^2 (1 + 2V(x_B^{fin})^2)}{1 + 2V(x_B^{fin})^2} - \frac{(p_{\Psi}^{fin})^2 (1 + 2V(p_B^{fin})^2)}{1 + 2V(p_B^{fin})^2}}.
\]
and on average over a Gaussian distributed input of coherent states with mean zero and standard deviation $n$ this yields

$$\bar{F}(\kappa, n, g_1, g_2) = \frac{2}{\sqrt{1 + 2V(x_{fin}^{f_B}) + 2n(1 - \frac{g_1\kappa}{2})^2} \sqrt{1 + 2V(p_{fin}^{f_B}) + 2n(1 + \frac{g_2\kappa}{2})^2}}$$

(7.16)

### 7.3 Comparison

Now, we want to compare Scheme I with Scheme II. For this purpose, both equations for the average fidelity are taken and optimized with respect to all gains and the interaction strength $\kappa > 0$ for different values of $n$. We note the symmetry of (7.15): because they are all positive, the terms in the denominator can be maximized all on their own, and we have $g_{3,\text{opt}} = -g_{1,\text{opt}}$ and $g_{4,\text{opt}} = -g_{2,\text{opt}}$ for Scheme I. The results for all optimal fidelities can be found in Fig. 7.3 and the corresponding gains and coupling strengths are depicted in Fig. 7.4. Clearly, Scheme I performs better than Scheme II for all values of $n$. 

---

**Figure 7.3:** The optimal average fidelity $\bar{F}$ is plotted for both schemes and for different values of $n$, the standard deviation of the Gaussian distribution of input states. The plot of Scheme I is the dotted line with stars, while the solid line with triangles is the result for Scheme II. The dash-dotted line with boxes shows the quantum benchmark for state transfer, (7.17).
Figure 7.4: Optimal gains and optimal $\kappa$ for the optimal average fidelities. (a) contains the optimal gains of Scheme I: $g_{1,\text{opt}} = -g_{3,\text{opt}}$ is the star-solid and $g_{2,\text{opt}} = -g_{4,\text{opt}}$ the box-dotted line. In (b), the star-solid line stands for $g_{1,\text{opt}}$, and the box-dotted one for $g_{2,\text{opt}}$ of Scheme II. (c) shows the optimal $\kappa$ in both schemes, the box-dotted line is for Scheme I and the star-solid line for Scheme II.
The intuitive expectation, that more light modes may introduce more unavoidable quantum noise to the system under consideration is not the entire truth. At the same time, we get the possibility to gain more information about the system by the additional measurement, which can be put to good use in additional degrees of freedom in the average fidelity, the additional gains.

The numerically calculated signs of the optimal gains match a heuristical understanding of the final variances in both schemes: They serve the purpose of minimizing these variances. This is necessary to get high fidelities, as can be seen from (7.15) and (7.16). The same argument also justifies in the retrospect the specific choice of signs in the usage of (2.39). Thinking about the signs reminds us in addition again of the postprocessing to remove the known phase in the teleported state’s displacement.

The quantum benchmark for transmission of coherent states [42, 43] is also plotted in Fig. 7.3. It is given by

\[ \bar{F}_{coh} = \frac{2n + 1}{4n + 1} \] (7.17)

and is the optimal fidelity that can be achieved by a classical strategy of measurement and according coherent state preparation. It turns out, that Scheme I is able to outperform any classical protocol for \( n < 3 \), i.e. for a small class of allowed input states. However, \( \kappa \approx 2 \) here, which is unfortunately too high for today’s experiments.

Nonetheless, we hope to contribute to the development of more sophisticated schemes in having shown how to improve the performance by using more feedback degrees of freedom. A first ansatz could for example be to also measure the first order light modes like \( p_{s/c,1,\pm} \) or \( p_{s/c,1,iii} \) and remove the corresponding noise this way. In consequence, the final states’ variances would get smaller and the fidelities higher.
8 Conclusions

In this diploma thesis, the off-resonant scattering interaction of linearly polarized laser light at room-temperature Cesium ensembles is considered [21]. Well defined forward scattering modes for light and certain collective spin variables of atoms allow for a simple and yet rather precise description of the atom-light interface in terms of only a few bosonic modes. The experimentally important states and interactions can be described in the Gaussian formalism.

The light modes get entangled with the atomic modes during the scattering, and by the subsequent interaction of the light with another distant ensemble, direct entanglement between the atomic modes can be induced [30, 31, 32, 33]. In the thesis at hand, two methods to enhance this protocol are investigated in the Gaussian setting, and the insights are employed in the application of quantum state teleportation between atomic ensembles.

The first method consists in exploiting light measurements and the information provided by their results. We show that certain measurements accelerate the entanglement generation between the atomic ensembles. However, feeding back their results into the system can in the Gaussian regime not be used to create more entanglement. This is the main contribution, which constitutes a general no-go-result in Gaussian entanglement generation using feedback. We remark that feedback can nevertheless be used for conditioning the output state in its linear version in a one-shot instead of the usual continuous form.

The second method is about the proper use of supplementary passive local operations during the interaction. We aim to create faster or more entanglement between the atomic ensembles by numerical optimization within a defined set of such operations. Our result is that antipodal static magnetic fields on the ensembles are an optimal configuration. They maximize the rate at which entanglement is created. Additional light operations on the other hand are of no further use.

In the teleportation application, we make use of such optimal magnetic fields. We compare two schemes and find the better to beat the quantum benchmark of state transfer in principle. It is therefore able to outperform any classical strategy. With more sophisticated schemes of this kind efficient quantum communication seems viable. This and emerging ideas for quantum information processing in atomic ensembles might make powerful quantum networks possible. By integrating non-Gaussian tools or even elements from discrete quantum information in hybrid systems [89], there is a promising future for quantum information with atomic ensembles.
A Realistic Transition Data

In this section the exemplary values for $\mu$ and $\nu$ are calculated for the realistic Hamiltonian, and the notation of Section 3.2 is used.

The ratio $R$ of passive and active part of the interaction, $\mu$ and $\nu$, can be calculated using equations for transition probabilities from $|g\rangle$ to $|s\rangle$ or reversely, that are obtained in the adiabatic elimination. Because we take into account the full multi-level structure of the excited-state manifold of Cesium, we have to consider allowed transitions to all excited states as intermediate steps. For the passive part, these are transitions to $|e_g\rangle = |F',m\rangle = 5,4$ and $|4,4\rangle$, and for the active part three excited states are involved, $|e_s\rangle = |5,3\rangle, |4,3\rangle$ and $|3,3\rangle$.

For the total transition probability of the active/passive part of the interaction, $\chi_{a/p}$, we have

$$\chi_a = \frac{1}{\Delta} \langle 4,4|5,1,3,1\rangle\langle 4,3|5,1,3,0\rangle + \frac{1}{\Delta + \Delta_{4,5}} \langle 4,4|4,1,3,1\rangle\langle 4,3|4,1,3,0\rangle + \frac{1}{\Delta + \Delta_{4,5} + \Delta_{3,4}} \langle 4,4|3,1,3,1\rangle\langle 4,3|3,1,3,0\rangle$$

$$= \frac{1}{\Delta} \sqrt{\frac{1}{90}} \left( -\sqrt{\frac{8}{45}} \right) + \frac{1}{\Delta + 251\text{MHz}} \left( -\sqrt{\frac{7}{120}} \right) \sqrt{\frac{21}{4160}} +$$

$$\frac{1}{\Delta + (251 + 201)\text{MHz}} \sqrt{\frac{7}{72}} \sqrt{\frac{7}{288}}$$

$$\chi_p = \frac{1}{\Delta} \langle 4,3|5,1,4,1\rangle\langle 4,4|5,1,4,0\rangle + \frac{1}{\Delta + \Delta_{4,5}} \langle 4,3|4,1,4,1\rangle\langle 4,4|4,1,4,0\rangle$$

$$= \frac{1}{\Delta} \sqrt{\frac{2}{5}} \left( -\sqrt{\frac{1}{10}} \right) + \frac{1}{\Delta + 251\text{MHz}} \sqrt{\frac{7}{120}} \sqrt{\frac{7}{30}}$$

$\langle F,m_F|F',1,m_{F'},q\rangle$ denote Clebsch-Gordan coefficients, which can be found in [90]. They appear, because coupling of the photon with spin 1 and polarization $q$ to the atom in the excited state $|F',m_{F'}\rangle$ involves coupling of the angular momenta to the ground state’s $|F,m_F\rangle$ angular momentum.

In addition, the single transition probabilities depend on the detuning with respect to the specific excited state under consideration. $\Delta_{4,5}$ is the hyperfine splitting between the states with $F'' = 4$ and $F' = 5$, and $\Delta_{3,4}$ the one between the states with $F'' = 3$ and $F' = 4$. 

Figure A.1: Relevant transitions in the Cs-multi-level structure. The active part is the one creating an excitation and a photon in $x$-polarization at the same time, it involves three transitions. The passive part creates a photon at the cost of an excitation, two transitions contribute here.

The ratio can then be calculated as

$$R = \frac{|\chi_p|}{|\chi_a|}.$$  

On the other hand, we have defined it as $R = \mu/\nu$, and therefore with $\mu = Z + 1/Z$ and $\nu = Z - 1/Z$

$$Z = \sqrt{\frac{|\chi_p| + |\chi_a|}{|\chi_p| - |\chi_a|}}.$$  

Again according to [90], $\Delta_{4,5} \approx 251\text{MHz}$ and $\Delta_{3,4} \approx 201\text{MHz}$. For a typical laser detuning of $\Delta = 855\text{MHz}$ with respect to the state with $F' = 5$, we get $Z \approx 2.5$ and $1/Z = 0.4$. From this, we can calculate the contribution of the active and the passive part of the interaction,

$$\mu = 1.45, \quad \nu = 1.05.$$
An effective Hamiltonian for both atomic ensembles can be introduced along the lines of and as a generalization of 3.2 \[32\]. We can see from Fig. B.1 that the allowed transitions now involve two sidebands, which are termed the upper/lower sideband with frequency $\omega_L \pm \Omega$ and creation operators $a_{us/l}^\dagger$. $\Omega$ is here the splitting of the ground states due to the Zeeman effect. We have, that $\Omega \ll \Delta$, such that $\mu$ and $\nu$ just depend on the multi-level structure and the detuning $\Delta$ and can be calculated as in Appendix A. With the same approximations, just such transitions are possible, that $H_{\text{int}} = H_1 + H_2$ with

$$H_1 = \mu b_1 a_{ls}^\dagger + \nu b_1 a_{us}^\dagger + \text{h.c.}, \quad H_2 = \mu b_2 a_{us}^\dagger + \nu b_2 a_{ls}^\dagger + \text{h.c.}.$$ 

Here, $b_{1/2}^\dagger$ is the creation operator of the usual collective bosonic mode of ensemble 1/2. The Hamiltonian can be understood by considering e.g. for ensemble 1 an excitation from the ground state $|4, 4\rangle$ to the excited state $|4, 3\rangle$ by the transition over $|3, 3\rangle$, $|4, 3\rangle$ and $|5, 3\rangle$. Then one atomic excitation, $b_1^\dagger$, and at the same time an upper sideband photon, $a_{us}^\dagger$, are created. If this process is described by the coupling $\nu$, in the second ensemble creation of an excitation, $b_2^\dagger$, by this transition involves the lower sideband, $a_{ls}^\dagger$.

It was noted in Section 3.3.3 that the useful (co)sine modes can be interpreted as linear combinations of upper/lower sideband modes. This concept is used here again and even extended to atomic (co)sine modes. We define

$$\begin{pmatrix} a_c \cr a_{ls}^\dagger \cr a_s \cr a_{us}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 0 & i & 0 & -i \end{pmatrix} \begin{pmatrix} a_{us} \cr a_{ls}^\dagger \cr a_s \cr a_{us}^\dagger \end{pmatrix}, \quad \begin{pmatrix} b_c \cr b_{ls}^\dagger \cr b_s \cr b_{us}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 0 & -i & 0 & i \end{pmatrix} \begin{pmatrix} b_1 \cr b_{ls}^\dagger \cr b_s \cr b_{us}^\dagger \end{pmatrix},$$

such that with the usual quadrature definitions

$$\begin{pmatrix} x_{L,c/us} \\ p_{L,c/us} \\ x_{L,s/l} \\ p_{L,s/l} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix} \begin{pmatrix} a_{c/us} \cr a_{c/us}^\dagger \cr a_{s/l} \cr a_{s/l}^\dagger \end{pmatrix}, \quad \begin{pmatrix} x_{A,c/1} \\ p_{A,c/1} \\ x_{A,s/2} \\ p_{A,s/2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix} \begin{pmatrix} b_{c/1} \\ b_{c/1}^\dagger \\ b_{s/2} \\ b_{s/2}^\dagger \end{pmatrix}.$$
Figure B.1: Two ensembles spin-polarized in \(x\)-direction. Left: magnetic field in \(+x\)-direction and according shift at ensemble I due to the Zeeman effect, right: field in \(-x\)-direction and according opposite shift at ensemble II. Dots indicate the ground state population. Note, that level-splittings are drawn here completely out of scale. (In a magnetic field of 1 Gauss, the Zeeman-shift of magnetic sublevels is \(\approx 10^5\) Hz, while the hyperfine and fine splitting is on the order of \(10^8\) Hz and \(10^{14}\) Hz, respectively.)

one can calculate the operator relations

\[
\begin{pmatrix}
a_{us}^\dagger & a_{us}^\dagger & a_{ls}^\dagger & a_{ls}^\dagger
\end{pmatrix}
\equiv
\frac{1}{2}
\begin{pmatrix}
1 & i & i & -1 \\
1 & -i & -i & -1 \\
i & i & -i & 1 \\
i & -i & i & 1
\end{pmatrix}
\begin{pmatrix}
x_{L,c} \\
p_{L,c} \\
x_{L,s} \\
p_{L,s}
\end{pmatrix},
\begin{pmatrix}
b_1 & b_1^\dagger & b_2 & b_2^\dagger
\end{pmatrix}
\equiv
\frac{1}{2}
\begin{pmatrix}
1 & i & -i & 1 \\
1 & -i & i & 1 \\
i & i & i & -1 \\
i & -i & -i & -1
\end{pmatrix}
\begin{pmatrix}
x_{A,c} \\
p_{A,c} \\
x_{A,s} \\
p_{A,s}
\end{pmatrix}.
\]

Also, we derive a relation between the new non-local and the local atomic modes,

\[
\begin{pmatrix}
x_{A,c} \\
p_{A,c} \\
x_{A,s} \\
p_{A,s}
\end{pmatrix}
\equiv
\frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
x_{A,1} \\
p_{A,1} \\
x_{A,2} \\
p_{A,2}
\end{pmatrix}
\equiv
U
\begin{pmatrix}
x_{A,1} \\
p_{A,1} \\
x_{A,2} \\
p_{A,2}
\end{pmatrix},
\tag{B.1}
\]

and find, that \(x_{A,c} = x_+, p_{A,c} = p_+, x_{A,s} = -p_- \) and \(p_{A,s} = x_-\). Inserting them in the definitions above, we derive

\[
H_{\text{int}} = \mu [x_{A,c}x_{L,c} + p_{A,c}p_{L,c} + x_{A,s}x_{L,s} + p_{A,s}p_{L,s}] + \\
\nu [x_{A,c}x_{L,c} - p_{A,c}p_{L,c} + x_{A,s}x_{L,s} - p_{A,s}p_{L,s}] \\
= (Z x_{A,c} x_{L,c} + \frac{1}{Z} p_{A,c} p_{L,c}) + (Z x_{A,s} x_{L,s} + \frac{1}{Z} p_{A,s} p_{L,s})
\]

Therefore the specific setup can be described by two independent sets of modes, which each can be treated like when no fields are applied at all.
C Measurements and Linear Feedback

As it is in this section important to distinguish between operators and numbers, we use "hats" for operators here. Consider a bipartite system of \(n + 1\) modes in the state \(\rho_{AB}\), of which \(n\) modes belong to subsystem \(B\). Let subsystem \(A\) be in the state given by \(\hat{x}_A\) and \(\hat{p}_A\). Then the effect of measurements on \(B\) of commuting observables \(\hat{R} \equiv \{\hat{r}_1, \ldots, \hat{r}_n\}, \hat{r}_i \in \{\hat{x}_i, \hat{p}_i\}\), and subsequent linear feedback of the results with so-called gains \(\hat{g}_x\) and \(\hat{g}_p\) on \(\hat{x}_A\) and \(\hat{p}_A\), respectively, is described by

\[
\hat{x}^{\text{fin}}_A = \hat{x}_A + \hat{R} \hat{g}_x^T, \quad \hat{p}^{\text{fin}}_A = \hat{p}_A + \hat{R} \hat{g}_p^T.
\]

This result can be applied in Chapter 7 by considering Bob’s mode as system \(A\) and the measured effective light modes as \(B\). \(A\) and \(\Psi\) may be considered as an additional system \(C\), about which we don’t make any statement at all.

The proof for this statement is sketched along the lines of \([25]\). For the measurement step, start with measurement results \(\vec{R} = (r_1, \ldots, r_n)\) appearing with probability \(p(\vec{R})\) and the according eigenvalue equations \(\hat{r}_i |r_i\rangle_B = r_i |r_i\rangle_B\). The conditional state of \(A\) after the measurement is given by

\[
\rho^{(1)}_A(\vec{R}) = B \langle r_1, \ldots, r_n | \rho_{AB} | r_1, \ldots, r_n \rangle_B / p(\vec{R}).
\]

The unitary displacement operator \(D_A \equiv \exp\{i \hat{R} \hat{g}_x^T \hat{x}_A\} \exp\{-i \hat{R} \hat{g}_p^T \hat{p}_A\}\) effecting \(D_A \hat{x}_A D_A^\dagger = \hat{x}_A + \hat{R} \hat{g}_x^T\) and \(D_A \hat{p}_A D_A^\dagger = \hat{p}_A + \hat{R} \hat{g}_p^T\) is then applied as a feedback operation,

\[
\rho^{(2)}_A(\vec{R}) = D_A^\dagger \rho^{(1)}_A(\vec{R}) D_A.
\]

In the ensemble average, we have

\[
\tilde{\rho}_A = \int d^n r p(\vec{R}) \rho^{(2)}_A(\vec{R})
= \int d^n r D_A^\dagger B \langle r_1, \ldots, r_n | \rho_{AB} | r_1, \ldots, r_n \rangle_B D_A
= \int d^n r B \langle r_1, \ldots, r_n | D_A^\dagger \rho_{AB} D_A | r_1, \ldots, r_n \rangle_B
= tr_B \{ D_A^\dagger \rho_{AB} D_A \}.
\]
for \( D_{AB} \equiv \exp\{i\hat{R}_x^{\tau}\hat{p}_A\} \exp\{-i\hat{R}_p^{\tau}\hat{x}_A\} \) acting on \( A \) and \( B \), because the \( \hat{r}_i \) mutually commute. Since

\[
\langle \hat{x}_A \rangle = \text{tr} \{ \hat{x}_A \hat{\rho}_A \} = \text{tr} \{ D_{AB} \hat{x}_A D_{AB}^\dagger \hat{\rho}_{AB} \} = \text{tr} \{ (\hat{x}_A + \hat{R}_x^{\tau}) \hat{\rho}_{AB} \}
\]

and an analogous calculation for \( \langle \hat{p}_A \rangle \) hold for all \( \hat{\rho}_{AB} \), we have the operator identity

\[
\hat{x}_{A_{\text{fin}}} = \hat{x}_A + \hat{R}_x^{\tau}, \quad \hat{p}_{A_{\text{fin}}} = \hat{p}_A + \hat{R}_p^{\tau}.
\]

Note, that averages have to be taken with respect to \( \hat{\rho}_{AB} \), the mutual state before measurement and feedback, in the new formalism.
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