Lecture "Quantum Information" WS 18/19 — Exercise Sheet #3

Problem 1: Dense coding.
Dense coding can be seen as the inverse protocol to teleportation. As in teleportation, Alice and Bob have free entanglement – Bell states $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ – at their disposal, but now, they want to use it to transmit classical information by sending quantum states as efficiently as possible, i.e., they want to transmit the maximum amount of classical information per qubit sent. Clearly, by encoding one classical bit into one qubit (e.g., as $|0\rangle$ and $|1\rangle$), they can transmit one classical bit per quantum bit sent. The goal is thus to do better.

1. Show that by acting on her part of $|\Phi^+\rangle$, Alice can transform the shared Bell state $|\Phi^+\rangle$ into any other Bell state.
2. Use this to set up a protocol where Alice can transmit two classical bits by sending only one quantum bit, by using the pre-shared Bell states. This protocol is called *dense coding*, or sometimes *super-dense coding.*
3. Use dense coding, together with teleportation, to show that both protocols are optimal given that shared entanglement is free – this is, we cannot send more classical bits per qubit transmitted, and teleportation of a qubit requires at least two classical bits to be sent (even if we use more complicated protocols sending larger amounts of data at once).

Problem 2: CHSH inequality I: Local hidden variable and no-signalling correlations.
Consider the scenario of the CHSH inequality. Let

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle \ .$$

Here, $a_x = \pm 1$ and $b_y = \pm 1$ are the outcomes obtained by Alice and Bob given an input (measurement setting) of x (on Alice's side) and y (on Bob's side). The measurement is described by some joint conditional probability distribution $P(a, b|x, y)$ [i.e., $\sum_{a,b} P(a, b|x, y) = 1$],

$$\langle a_x b_y\rangle = \sum_{a,b} a\, b\, P(a, b|x, y) \ .$$

1. A local hidden variable (LHV) distribution is of the form
$$P(a, b|x, y) = \sum_\lambda p_\lambda P^A_\lambda(a|x) P^B_\lambda(b|y) \ .$$
Use this to derive the bound $|\langle C\rangle| \leq 2$. (This should be done by explicitly using the form of $P(a, b|x, y)$, not by making any intuitive assumptions about LHV distributions. *Hint:* It can be helpful – though not necessary – to make P^A_λ and P^B_λ deterministic by introducing a new random variable λ.) Which property of $P(a, b|x, y)$ allows to obtain this bound?
2. A non-signalling distribution is a distribution which does not allow for communication between Alice and Bob, i.e., Alice's marginal distribution $P^A(a|x) = \sum_b P(a, b|x, y)$ does not depend on Bob's input y, and vice versa. Show that non-signalling distributions can obtain the maximum possible value $|\langle C\rangle| = 4$.
3. Give a distribution $P(a, b|x, y)$ which violates no-signalling.

Problem 3: CHSH inequality II: Tsirelson's bound.
Tsirelson's inequality bounds the largest possible violation of the CHSH inequality in quantum mechanics (namely $2\sqrt{2}$). To this end, let a_0, a_1, b_0, b_1 be Hermitian operators with eigenvalues ± 1, so that

$$a_0^2 = a_1^2 = b_0^2 = b_1^2 = \mathbb{1} \ .$$

Here, a_0 and a_1 describe the two measurements of Alice, and b_0 and b_1 those of Bob; in particular, this means that Alice's and Bob's measurements commute, i.e. $[a_x, b_y] = 0$ for all $x, y = 0, 1$. Define

$$C = a_0 b_0 + a_1 b_0 + a_0 b_1 - a_1 b_1 \ .$$

1. Determine C^2.
2. The norm of a bounded operator M is defined by

$$\|M\| = \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} ,$$

that is, the norm of M is the maximum eigenvalue of $\sqrt{M^\dagger M}$. Verify that the norm has the properties

$$\|MN\| \leq \|M\|\|N\| ,$$
$$\|M + N\| \leq \|M\| + \|N\| .$$

3. Find an upper bound on the norm $\|C^2\|$.
4. Show that for Hermitian operators $\|C^2\| = \|C\|^2$. Use this to obtain an upper bound on $\|C\|$.
5. Explain how this inequality gives a bound on the maximum possible violation of the CHSH inequality in quantum mechanics. This is known as Tsirelson's bound, or Tsirelson's inequality.

Problem 4: LOCC protocols.
Suppose $|\psi\rangle$ can be transformed to $|\phi\rangle$ by LOCC. A general LOCC protocol can involve an arbitrary number of rounds of measurement and classical communication. In this problem, we will show that any LOCC protocol can be realized in a single round with only one-way communication, i.e., a protocol involving just the following steps: Alice performs a single measurement described by measurement operators K_j, sends the result j to Bob, and Bob performs a unitary operation U_j on his system.
The idea is to show that the effect of any measurement which Bob can do can be simulated by Alice (with one small caveat) so all Bob's actions can actually be replaced by actions by Alice.

1. First, suppose that Bob performs a measurement with operators $M_j = \sum_{kl} M_{j,kl}|k\rangle_B\langle l|_B$ on a pure state $|\psi\rangle_{AB} = \sum \lambda_l |l\rangle_A |l\rangle_B$, with the resulting state denoted as $|\psi_j\rangle$. Now suppose that Alice performs a measurement with operators $N_j = \sum_{kl} M_{j,kl}|k\rangle_A\langle l|_A$ on a pure state $|\psi\rangle$, with resulting state denoted as $|\phi_j\rangle$. Show that there exist unitaries U_j on system A and V_j on system B such that $|\psi_j\rangle = (U_j \otimes V_j)|\phi_j\rangle$.
2. Use this to explain how any multi-round protocol can be implemented with one measurement done by Alice followed by a unitary operation done by Bob which depends on Alice's outcome.

Problem 5: Entanglement conversion of multiple copies.
Consider the problem of converting a state $|\chi\rangle = \sqrt{\frac{3}{4}}|00\rangle + \sqrt{\frac{1}{4}}|11\rangle$ to the Bell state $|\Phi^+\rangle = \sqrt{\frac{1}{2}}|00\rangle + \sqrt{\frac{1}{2}}|11\rangle$. As we have seen in the lecture, the maximum success probability for this conversion can be determined using the majorization criterion.

1. Determine the maximum success probability P_1 for converting $|\chi\rangle$ into $|\Phi^+\rangle$.
2. Show that there is a protocol which takes three copies of $|\chi\rangle$ and produces into 3 copies of $|\Phi^+\rangle$ with probability $p_3 = \frac{1}{8}$ and 2 copies with $p_2 = \frac{5}{8}$, respectively. What is the average yield P_3 of $|\Phi^+\rangle$ per copy of $|\chi\rangle$ used?
3. Show that by using 2 copies, the average yield does not improve as compared to one copy.
4. Show that the protocol of 2. is optimal.
5. Find the optimal protocol for converting 2 or 3 copies of a state $|\chi_t\rangle = \sqrt{1-t}|00\rangle + \sqrt{t}|11\rangle$ to Bell states. What is the maximum gain in the yield of $|\Phi^+\rangle$, i.e., P_2/P_1 and P_3/P_1?

(*Note:* While straightforward, it might be helpful to use a computer algebra system for solving the systems of equations.)