

## Lecture “Quantum Information” WS 18/19 — Exercise Sheet #2

### Problem 1: Measurements and filtering

Suppose that the initial state of system  $AB$  is

$$|\phi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle.$$

The goal is to obtain a maximally entangled state  $|\phi_{0.5}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$  with some probability after a measurement on system  $A$ .

1. Show that the operators  $\Pi_0 = (|0\rangle\langle 0| + \sqrt{\gamma}|1\rangle\langle 1|)_A \otimes \mathbb{1}_B$  and  $\Pi_1 = \sqrt{1-\gamma}|1\rangle\langle 1|_A \otimes \mathbb{1}_B$ , with  $0 \leq \gamma \leq 1$ , define a POVM measurement. (Note that these describe measurements carried out on Alice’s side only!)
2. Determine the outcome probabilities and the post-measurement states for each measurement outcome.
3. Find a value  $\gamma$  such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.

### Problem 2: Quantum channels.

In this problem, we will study some commonly appearing quantum channels. In addition to the problems listed, verify for each channel that it is a CPTP map (completely positive trace preserving map) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as

$$\mathcal{E}(\rho) = (1-p)\rho + pZ\rho Z.$$

Show that the action of the dephasing channel on the Bloch vector is

$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z),$$

i.e., it preserves the component of the Bloch vector in the  $Z$  direction, while shrinking the  $X$  and  $Y$  component.

2. *Amplitude damping channel.* The amplitude damping channel is given by the Kraus operators

$$\Pi_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad \Pi_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|,$$

where  $0 \leq \gamma \leq 1$ . Here,  $\Pi_0$  describes a decay from  $|1\rangle$  to  $|0\rangle$ , and  $\gamma$  corresponds to the decay rate.

- (a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

where  $0 \leq p \leq 1$  and  $\eta$  is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

- (b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel  $\mathcal{E}_1$  with parameter  $\gamma_1$  and consider another amplitude damping channel  $\mathcal{E}_2$  with parameter  $\gamma_2$ . Show that the composition of the channels,  $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$ ,  $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$ , is an amplitude damping channel with parameter  $1 - (1-\gamma_1)(1-\gamma_2)$ . Interpret this result in light of the interpretation of the  $\gamma$ ’s as a decay probability.
3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z.$$

Show that the output of this channel is the maximally mixed state for any input,  $\mathcal{E}(\rho) = \frac{1}{2}\mathbb{1}$ .

*Hint:* Represent the density operator as  $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$  and apply the commutation rules of the Pauli operators.

**Problem 3: Purifications.**

1. Consider the following three ways of expressing the maximally mixed state as an ensemble:

$$\frac{1}{2}\mathbb{1} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|.$$

- (a) Construct purifications for all three ensemble decompositions, such that the corresponding ensemble is obtained upon measuring the purifying system in the computational basis.
- (b) Show that all those purifications can be transformed into each other by acting with a unitary on the purifying systems, and explicitly construct this unitary (remember that you might have to pad with zeros).
- (c) Give POVM measurements which realize each of the three ensemble decompositions by measuring the maximally entangled state  $(|00\rangle + |11\rangle)/\sqrt{2}$ .
2. Consider two ensemble decompositions

$$\sum p_i |\psi_i\rangle\langle\psi_i| = \sum q_j |\phi_j\rangle\langle\phi_j| = \rho$$

of the same density matrix  $\rho$ . We will prove that in that case, the ensembles are related via

$$\sqrt{p_i} |\psi_i\rangle = \sum_j u_{ij} \sqrt{q_j} |\phi_j\rangle \tag{1}$$

with  $(u_{ij})$  a unitary (after possibly padding the decompositions with zeros). To start with, restrict to the case where  $|\phi_i\rangle$  is an eigenbasis, and define

$$u_{ij} = \langle\phi_j|\psi_i\rangle \sqrt{p_i/q_j}.$$

- (a) Show that  $u_{ij}$  fulfils Eq. (1).
- (b) Show that  $(u_{ij})$  has orthogonal columns.
- (c) Show that by padding the  $|\phi_i\rangle$  decomposition with zeros, we can make  $(u_{ij})$  unitary.
- (d) Now consider the general case, where  $|\phi_i\rangle$  is not necessarily an eigenbasis.

*Hint:* Connect the two ensembles by going through the eigenvalue decomposition of  $\rho$ .

**Problem 4: Schmidt decomposition.**

Find the Schmidt decomposition of the following states:

$$\begin{aligned} |\psi_1\rangle &= \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \\ |\psi_2\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle}{\sqrt{2}} \\ |\psi_3\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}{2} \\ |\psi_4\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle}{2} \\ |\psi_5\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{3}} \end{aligned}$$