Lecture “Quantum Information” WS 18/19 — Exercise Sheet #1

Problem 1: Pauli matrices.
The following matrices, written in the *computational basis* $\{|0\rangle, |1\rangle\}$, are called the *Pauli matrices*:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

1. Show that the Pauli matrices are all hermitian, unitary, square to the identity, and different Pauli matrices anticommute.

2. It is common to label the Pauli matrices together with the identity matrix $\mathbb{1}$ as $\sigma_0 = \mathbb{1}$, $\sigma_1 = X$, $\sigma_2 = Y$ and $\sigma_3 = Z$. Show that $\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$ for all $i, j \in \{0, ..., 3\}$. Here, δ_{ij} is a Kronecker delta function, and the *trace* $\mathrm{tr}(M) = \sum_i \langle i|M|i\rangle = \sum_i M_{ii}$ is the sum of all diagonal elements of M. (Recall that the trace only depends on the eigenvalues and is thus basis independent.)

3. Write each operator X, Y and Z using bra-ket notation with states from the computational basis.

4. Find the eigenvalues e_i and eigenvectors $|v_i\rangle$ of the Pauli matrices, and write them in their diagonal form $e_1|v_0\rangle\langle v_0| + e_1|v_1\rangle\langle v_1|$.

5. Determine the measurement operators E_n corresponding to a measurement of the Y observable. For a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, determine the probabilities for the different outcomes for a measurement of the Y observable, and find the corresponding post-measurement states.

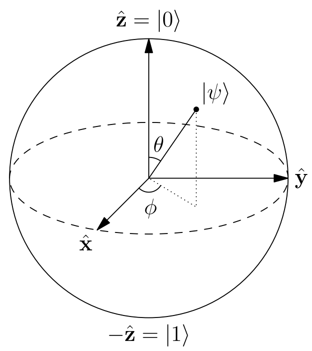
6. Write all tensor products of Pauli matrices (including the identity) as 4×4 matrices.

Problem 2: Bloch sphere for pure states.

1. Show that any pure qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, can be written as

$$|\psi\rangle = e^{i\alpha}\big[\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\big] , \qquad (1)$$

where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$, and $e^{i\alpha}$ is an irrelevant global phase. The angles θ and ϕ can be interpreted as spherical coordinates describing a point on a sphere, the so-called *Bloch sphere*, as shown in the figure to the right.



(Source: Wikipedia)

2. Determine the Bloch sphere angles θ and ϕ for the eigenstates of the Pauli X, Y, and Z operator, and locate the corresponding points on the Bloch sphere.

3. Show that Eq. (1) implies that

$$|\psi\rangle\langle\psi| = \tfrac{1}{2}(\mathbb{1} + \vec{v}\cdot\vec{\sigma}) \ \text{ with } \vec{v} \in \mathbb{R}^3 \text{ and } |\vec{v}| = 1 , \qquad (2)$$

(i.e., $\vec{v}$ is a vector on the unit sphere in $\mathbb{R}^3$), where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^3 v_i\sigma_i$ with σ_i as in Problem 1. (You should find that $\vec{v}$ is exactly the point on the Bloch sphere with spherical coordinates in θ and ϕ, just as in the figure.)

Note: The vector $\vec{v}$ is called the *Bloch sphere representation* of the state $|\psi\rangle$.

4. Show that the expectation value of the Pauli operators is $\langle\psi|\sigma_i|\psi\rangle = v_i$; i.e., $|\psi\rangle$ desribes a spin which is polarized along the direction $\vec{v}$.
(*Note:* This is particularly easy to show if you use that $\langle\psi|O|\psi\rangle = \mathrm{tr}[|\psi\rangle\langle\psi|\,O]$ together with Eq. (2) and $\mathrm{tr}[\sigma_i\sigma_j] = 2\delta_{ij}$, but can of course also be derived from Eq. (1).)

5. Show that for any state $|\psi\rangle$ with corresponding Bloch vector $\vec{v}$, the state $|\phi\rangle$ orthogonal to it, i.e. with $\langle\psi|\phi\rangle = 0$ (for qubits, i.e., in $\mathbb{C}^2$, this state is uniquely determined up to a phase!), is described by the Bloch vector $-\vec{v}$, i.e., it is located at the opposite point of the Bloch sphere.

Problem 3: Bell states.

1. Show that the singlet state
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} - |10\rangle_{AB}\right)$$
is invariant under joint rotations by the same 2×2 unitary U, i.e.,
$$|\Psi^-\rangle = (U\otimes U)|\Psi^-\rangle$$
for any unitary matrix U, $U^\dagger U = \mathbb{1}$.

2. Show that this implies that if we measure the spin in any direction $\vec{v}$, $|\vec{v}| = 1$ – this measurement is described by the measurement operator $S_{\vec{v}} = \sum_{i=1}^{3} v_i\sigma_i$ – we obtain perfectly random and opposite outcomes.

   (*Hint:* An elegant way of doing so is to first show that any $S_{\vec{v}}$ has the same eigenvalues as the Z matrix and therefore can be rotated to it, i.e., there exists a $U_{\vec{v}}$ s.th. $U_{\vec{v}}S_{\vec{v}}U_{\vec{v}}^\dagger = Z$.)

3. Determine the states
$$\begin{aligned}
&(X\otimes\mathbb{1})|\Psi^-\rangle\ , \qquad &&(\mathbb{1}\otimes X)|\Psi^-\rangle\ ,\\
&(Y\otimes\mathbb{1})|\Psi^-\rangle\ , \qquad &&(\mathbb{1}\otimes Y)|\Psi^-\rangle\ ,\\
&(Z\otimes\mathbb{1})|\Psi^-\rangle\ , \qquad &&(\mathbb{1}\otimes Z)|\Psi^-\rangle\ .
\end{aligned}$$

   How can we understand from 1. that they are pairwise equal?

   Note: Together with $|\Psi^-\rangle$, these are known as the four *Bell states.*

Problem 4: Bloch sphere for mixed states.

1. Prove that any hermitian 2×2 matrix ρ with $\operatorname{tr}\rho = 1$ can be written as
$$\rho = \frac{1}{2}(\mathbb{I} + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbb{I} + r_x X + r_y Y + r_z Z)\ ,$$
where, as in Problem 2, $\vec{\sigma}$ is the vector consisting of the three Pauli matrices X, Y, Z, and $\vec{r}\in\mathbb{R}^3$ is the *Bloch vector* of the system.

2. Show that ρ is a density operator (i.e., $\rho \geq 0$) if and only if $|\vec{r}| \leq 1$.

3. Prove that ρ is pure if and only if $|\vec{r}| = 1$.

   Note: These results show that the surface of the Bloch sphere corresponds to all pure states and its interior corresponds to all mixed states.

4. Give the Bloch vectors corresponding to the following states and draw them on the Bloch sphere:
$$\text{(a) } \rho_a = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\ ;\quad \text{(b) } \rho_b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ ;\quad \text{(c) } \rho_c = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\ .$$