Numerical solution of nonlinear Maxwell's equations in the PSC [1] (Plasma Simulation Code)

Patrick Böhl†, Ben King†, Hartmut Ruhl†

†Ludwig-Maximilians-Universität München

Idea

We extend the existing PSC, a Particle-in-Cell code, which was developed to simulate a plasma under the influence of very high external fields, by a Finite-Differences-Time-Domain (FDTD) solver for Maxwell's equations of the general form

$$\begin{aligned}\frac{\partial \vec{E}(\vec{x},t)}{\partial t} &= \vec{\nabla} \times \vec{H}(\vec{x},t) - \mathcal{N}(\vec{E}(\vec{x},t), \vec{H}(\vec{x},t)) \\ \frac{\partial \vec{H}(\vec{x},t)}{\partial t} &= -\vec{\nabla} \times \vec{E}(\vec{x},t)\end{aligned} \tag{1}$$

where $\mathcal{N}(\vec{E}(\vec{x},t), \vec{H}(\vec{x},t))$ is an arbitrary, non-linear function of the electric field $\vec{E}$ and the magnetic field $\vec{H}$ and their spatial and temporal derivatives.

Method

In standard FDTD-schemes, equations (1) are discretised using a staggered Yee-grid, where the field components are defined at different points on the grid. Inspecting (1) shows that the non-linear part $\mathcal{N}$ has to be evaluated at the grid points where the E-field ist defined. Depending on $\mathcal{N}$, one can achieve this by interpolation, but also staggered or unstaggered, colocated grids are possible [2].

Example: Kerr-nonlinear medium

One popular example of a nonlinearity is given by the instantanous Kerr-Effect wich is described by a polarisation

$$\vec{P}(\vec{x},t) = \chi^{(3)} \left|\vec{E}(\vec{x},t)\right|^2 \vec{E}(\vec{x},t)$$

where $\chi^{(3)}$ is the nonlinear Kerr susceptibility and the corresponding polarisation current is given by [3]:

$$\vec{J}_{\text{pol}} = \partial_t \vec{P} = \chi^{(3)} \partial_t \left|\vec{E}(\vec{x},t)\right|^2 \vec{E}(\vec{x},t)$$

The equations we want to solve therefore read:

$$\begin{aligned}\partial_t \vec{E}(\vec{x},t) &= \vec{\nabla} \times \vec{H}(\vec{x},t) - \chi^{(3)} \partial_t \left|\vec{E}(\vec{x},t)\right|^2 \vec{E}(\vec{x},t) \\ \partial_t \vec{H}(\vec{x},t) &= -\vec{\nabla} \times \vec{E}(\vec{x},t)\end{aligned} \tag{2}$$

Discretising (2) in time by approximating

$$\partial_t \vec{E}(\vec{x},t) \approx \frac{1}{\Delta t}\left[\vec{E}^{n+1}(\vec{x}) - \vec{E}^{n}(\vec{x})\right]$$

where the index n denotes the actual timestep (analogously for $\vec{H}$), we arrive at an implicit nonlinear equation for $\vec{E}^{n+1}$ which is solved up to machine precision. The coupled system is then subjected to a standard leapfrog algorithm.

1D Reference solution

There are only very few analytical solutions known to coupled systems of nonlinear partial differential equations. Even fewer if one wants to consider more than one-dimensional systems. Therefore as a benchmark, we choose a reference solution for a Kerr-model in one dimension, given in [4]. The considered set of equations read:

$$\begin{aligned}\partial_t E(x,t) &= \frac{1}{1+3\chi^{(3)}E(x,t)^2}\partial_x H(x,t) \\ \partial_t H(x,t) &= \partial_x E(x,t)\end{aligned} \tag{3}$$

The solution for (3) is given by

$$E(x,t) = F\left(x - \frac{1}{\sqrt{1+3\chi^{(3)}E(x,t)^2}}\cdot t\right) \quad , \tag{4}$$

where F is an arbitrary, smooth function and the H-field is then given by a formal power series in F, which we do not need here.

We take the simulation box to be 10 cm and a resolution of 2000 points for a colocated, non-staggered grid. In [4] the authors show, that for a reasonable choice of parameters, the existence of a unique solution to (4) is guaranteed for a simulation time of 50 ps.

References

[1] Plasma simulation code. `http://www.plasma-simulation-code.net/`.

[2] A. Taflove and S. C. Hagness. *Computational Electrodynamics: The Finite-Difference Time-Domain Method, 3rd ed.* Artech House Inc, 2005.

[3] J. H Greene and A. Taflove. General vector auxiliary differential equation finite-difference time-domain method for nonlinear optics. *Optics express*, 14(18):8305–10, September 2006.

[4] M. Pototschnig et al. Time-Domain Simulations of the Nonlinear Maxwell Equations Using Operator-Exponential Methods. *IEEE Transactions on Antennas and Propagation*, 57(2):475–483, February 2009.

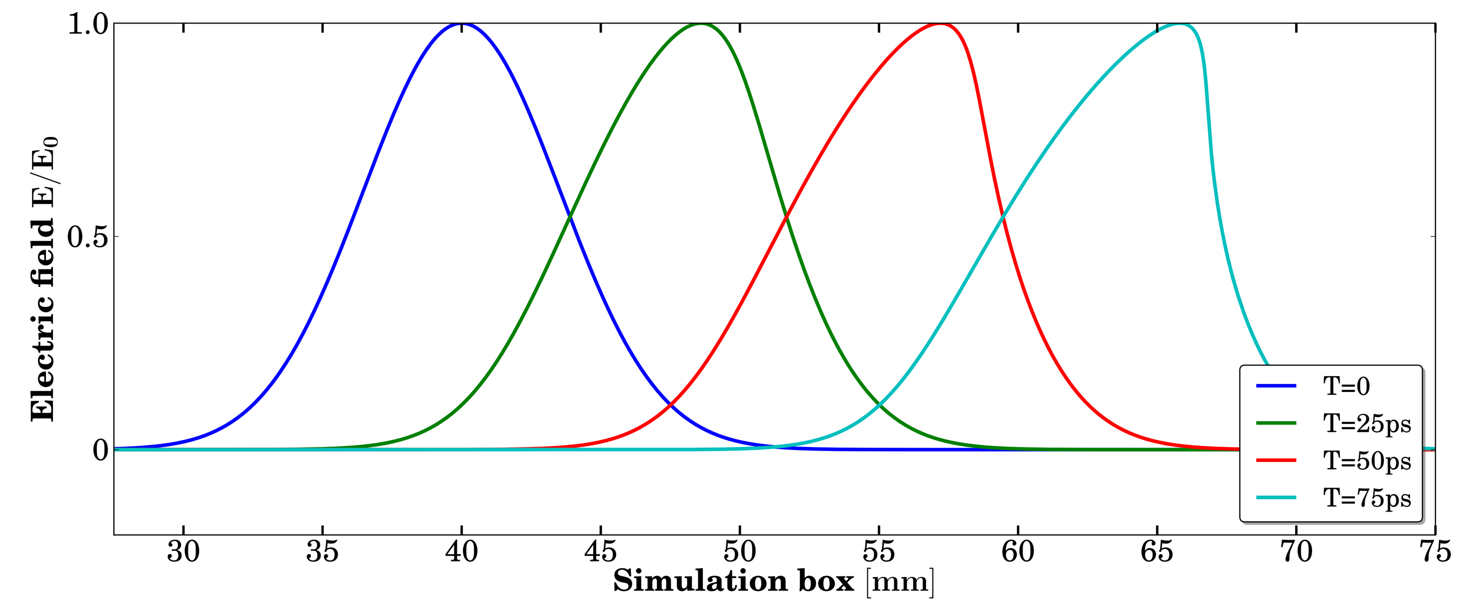
[5] W. Heisenberg and H. Euler. Consequences of Dirac's theory of positrons. *Z.Phys.*, 98:714–732, 1936.

[6] B. King, A. di Piazza, and C. H. Keitel. A matterless double slit. *Nature Photonics*, 4:92–94, February 2010.

Numerical results

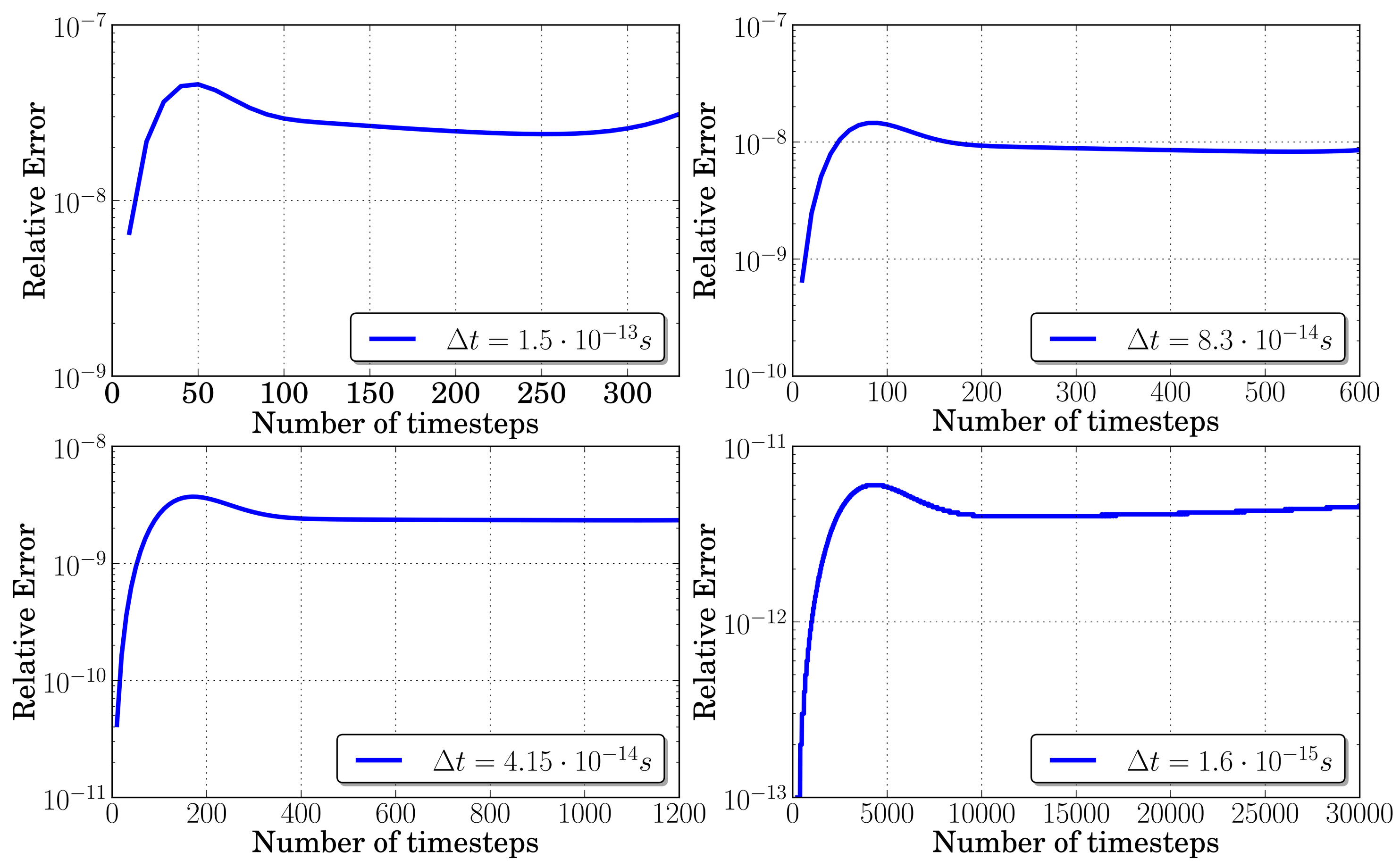
The initial condition is a Gaussian profile: $E(x,0) = E_0 \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]$.

The time evolution is shown in the following graph:



We see a steepening of the pulse and eventually (not shown) the wave will "break" similar to a shallow water wave.

The relative error, which is defined as $||E - E_{ref}||/||E_{ref}||$ with $||E|| := (\sum_{\text{grid}} E^2)^{1/2}$ being the sum over all E-values on the grid is shown for different values of Δt is shown in the following graph:



The simulation time is always taken to be 50 picoseconds.

Application: QED

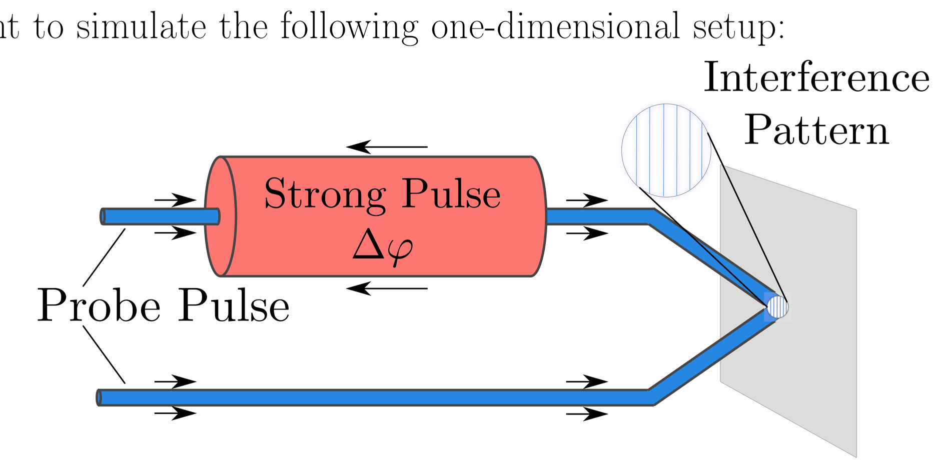
In quantum electrodynamics, the theory of electrons, positrons and photons, it is possible in the limit $\omega/m \ll 1$ (which means that the energy of the photons is much smaller than the rest mass of the electron) to arrive at an effective theory only for photons, by "integrating out" the fermions. This was first done by Euler and Heisenberg [5]. As a consequence, one obtains corrections to the classical Maxwell's equations, which can be perturbatively expanded in $\sqrt{\alpha}E/E_{crit}$ where $\alpha = 1/137$ is the fine structure constant and $E_{crit} \approx 1.3 \cdot 10^{18}\ \frac{\text{V}}{\text{m}}$ is the Schwinger field. To lowest order, these read:

$$\begin{aligned}-\partial_t \vec{E} + \vec{\nabla} \times \vec{B} = \frac{\alpha}{45 E_{crit}^2}\Big[& 8\left[2(\vec{B}\cdot\vec{\nabla}\times\vec{E})\cdot\vec{E} + 2(\vec{E}\cdot\partial_t\vec{E})\vec{E} - (\vec{B}^2 - \vec{E}^2)\partial_t\vec{E}\right] \\ & + 28\left[(\partial_t\vec{E}\cdot\vec{B})\vec{B} - (\vec{E}\cdot\vec{\nabla}\times\vec{E})\vec{B} - (\vec{E}\cdot\vec{B})\vec{\nabla}\times\vec{E}\right] \\ & + 8\left[\vec{\nabla}(\vec{B}^2 - \vec{E}^2)\times\vec{B} + (\vec{B}^2 - \vec{E}^2)(\vec{\nabla}\times\vec{B})\right] \\ & - 28\left[\vec{\nabla}(\vec{E}\cdot\vec{B})\times\vec{E} + (\vec{E}\cdot\vec{B})\vec{\nabla}\times\vec{E}\right]\Big]\end{aligned} \tag{5}$$

The next generation of high power optical Laser facilities will not be able to reach the critical field strength, however such effects are predicted to be observable [6].

Future work

As a first aim, we want to simulate the following one-dimensional setup:



Two identical X-ray probe pulses propagate over the same distance, but one of them runs through a counterpropagating, optical, strong pulse. The interaction between the probe and the strong pulse leads to a measurable phase shift $\Delta\varphi$ according to (5). This phase shift should be visible in the interference pattern of the former identical pulses.

For the future, we want to extend these calculations to more than one dimension and also see the full backreaction of a system of electrons and positrons to these nonlinearities.