A Rigorous Derivation of Electromagnetic Self-force

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The Question: How does a charged particle's own electromagnetic field affect its motion?

But what is a ``charged particle''?

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But rigidity is incompatible with special relativity

Dirac: a true point particle

But this results in infinities

Over the last 100 years, many analyses based on either rigidity or point particles have appeared. All have derived the Abraham-Lorentz-Dirac equation,

$$ma_\alpha = qF^{\text{ext}}_{\alpha\beta}u^\beta + \frac{2}{3}q^2\left(g_{\alpha\beta} + u_\alpha u_\beta\right)\dot{a}^\beta$$

This equation admits runaway solutions. They are usually eliminated by making the substitution

$$\dot{a}_\beta = u^\gamma \nabla_\gamma \left(\frac{q}{m}F^{\text{ext}}_{\beta\sigma}u^\sigma\right)$$

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Problems facing a rigorous treatment

1. Since point particles and rigid bodies do not work, the particle must be modeled in some full continuum description.

What matter model to choose?
Will the results be independent of this choice?

2. Do we keep the body at finite size or take a limit to zero size?

If kept at finite size, all of the internal degrees of freedom will affect its motion, and it is not obvious how to define e.g. center of mass.

If a limit to zero size is taken in the usual way, all of the difficulties of the point particle description are recovered.

Solutions

1. Assume only that the matter model conserves total (matter plus EM) stress-energy, so that the results hold for bodies made of any matter with this basic property.

2. Consider a *modified* point particle limit, wherein not only the size but also the charge and mass are taken to zero. (This limit is the key idea of our approach.)

The Modified Point Particle Limit

The usual point particle limit is λ→0 for a family of charge densities of the form

$$\rho(\lambda; t, x^i) = \lambda^{-3}\tilde{\rho}(t, \frac{x^i - z^i(t)}{\lambda})$$

As the size goes to zero the total charge is fixed and the shape is preserved.

We want instead

$$\rho(\lambda; t, x^i) = \lambda^{-2}\tilde{\rho}(\lambda; t, \frac{x^i - z^i(t)}{\lambda})$$

The total charge decreases linearly and the shape is preserved asymptotically

In the modified point particle limit the body <u>disappears</u> at λ=0.
But the worldline $z^i(t)$ is where it “disappeared from” and represents its leading-order motion.

Our assumptions

We assume that there is a one-parameter-family of solutions $\{F_{\mu\nu}(\lambda), J^\mu(\lambda), T^M_{\mu\nu}(\lambda)\}$ to the "Maxwell and matter equations",

$$\boxed{\begin{array}{ll} \nabla^\nu F_{\mu\nu} = 4\pi J_\mu & \\ & \nabla^\nu\left(T^M_{\mu\nu} + T^{EM}_{\mu\nu}\right) = 0 \\ \nabla_{[\mu} F_{\nu\rho]} = 0 & \end{array}}$$

$$T^{EM}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

such that

$$J^\mu(\lambda, t, x^i) = \lambda^{-2}\tilde{J}^\mu(\lambda, t, [x^i - z^i(t)]/\lambda)$$

$$T^M_{\mu\nu}(\lambda, t, x^i) = \lambda^{-2}\tilde{T}_{\mu\nu}(\lambda, t, [x^i - z^i(t)]/\lambda)$$

with $\tilde{J}^\mu$ and $\tilde{T}_{\mu\nu}$ smooth, and

$$F_{\mu\nu}(\lambda) = F^{\text{ext}}_{\mu\nu}(\lambda) + F^{\text{ret}}_{\mu\nu}(\lambda)$$

Our goal is to determine what worldlines z^i are allowed as well as compute perturbative corrections

where $F^{\text{ret}}_{\mu\nu}(\lambda)$ is the retarded solution associated with $J^\mu(\lambda)$ and $F^{\text{ext}}_{\mu\nu}$ is smooth.

Aside: Why One-parameter Families?

Two viewpoints on perturbation theory:

Viewpoint #1: drop terms that one expects to be numerically small in situations of interest

$$f(R;x) = f(0;x) + \partial_R f(0;x) R + O(\cancelto{\text{small}}{R^2})$$

Viewpoint #2: Just compute f and some number of derivatives at R=0.

Note the trivial fact that even viewpoint #1 is "taking the body to zero size"

These viewpoints are equivalent as long as viewpoint #1 does not accidentally make inconsistent approximations, as can happen in complicated derivations where many such small terms are dropped. Viewpoint #2 just follows the rules of math.

In a such a controversial problem as self-force, it pays to take viewpoint #2. If the assumptions are reasonable and the math correct, the result holds.

Review of The Setup

At finite λ, we have a finite-sized blob of classical charged matter

As λ→0, the size, mass, and charge (and hence self-energy) of the blob go to zero

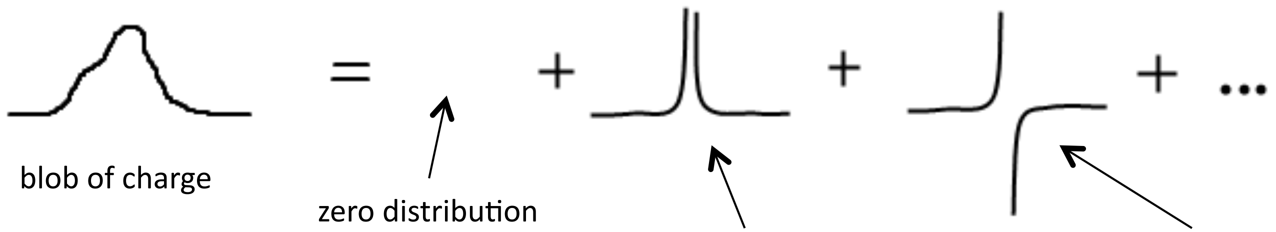
We will do perturbation theory (i.e., Taylor expand in λ) to approximate the finite λ body with properties at λ→0.

The results should accurately describe a finite-size body that is small compared to its surroundings and whose self-energy is not too great.

(The results can be applied to electrons only insofar as a classical description of the electron can be justified.)

The “Far zone” viewpoint

Our one-parameter family takes the body to zero size and therefore results in a *distributional* description in perturbation theory.



finite λ

λ^0

λ^1

λ^2

blob of charge

zero distribution

delta function
times total charge

derivative of delta function
times dipole moment

Using (distributional) conservation of T and J at O(λ), we derive

$$J^{(1)\mu} = q u^\mu \delta(x^i - z^i(t)) \frac{d\tau}{dt} \qquad m u^\nu \nabla_\nu u_\mu = q u^\nu F^{\mathrm{ext}}_{\mu\nu}$$

The worldline z^i obeys the Lorentz force law, and the field of the body is that of a point particle! Thus the Coulomb field and force (Lienard-Viechert field and Lorentz force) force emerge as approximate descriptions of a small body.

Note that the electromagnetic self-energy makes a finite contribution to the mass eq.(22) now yield

$$D_{T^{\rm self}}[f] = \lambda \int \tilde{T}^{\rm self}_{\mu\nu}(\lambda, t, \bar{x}^i) f^{\mu\nu}(t, z^i(t) + \lambda \bar{x}^i) \sqrt{-g} dt d^3\bar{x} \ . \tag{38}$$

If $\tilde{T}^{\rm self}_{\mu\nu}$ were of compact support in $\bar{x}^i$, we could straightforwardly take the limit as $\lambda \to 0$ inside the integral, as we did in eq.(22). Nevertheless, using the facts that (1) the integrand is smooth in $\bar{x}^i$, (2) the test tensor field $f^{\mu\nu}$ is of compact support in x^i (and, thus, is of compact support in α) and (3) for α in a compact set, $\tilde{T}^{\rm self}_{\mu\nu}$ is bounded by $C\beta^4$ for some constant C, it is not difficult to show that for $\lambda \leq \lambda_0$

$$|\tilde{T}^{\rm self}_{\mu\nu}(\lambda, t, \bar{x}^i)||f^{\mu\nu}(t, z^i(t) + \lambda \bar{x}^i)| \leq \frac{K}{\bar{r}^4 + 1} \tag{39}$$

where K is a constant (i.e., independent of λ and $\bar{x}^i$). Since $1/[\bar{r}^4 + 1]$ is integrable with respect to $\sqrt{-g} d^3\bar{x}$, the dominated convergence theorem then allows us to take the limit as $\lambda \to 0$ inside the integral in this case as well, and we obtain

$$D_{T^{\rm self}}[f] = \lambda \int dt \frac{d\tau}{dt} f^{\mu\nu}(t, z^i(t)) \int d^3\bar{x} \sqrt{g_3} \tilde{T}^{\rm self}_{\mu\nu}(0, t, \bar{x}^i) + O(\lambda) \ , \tag{40}$$

The “Near zone” viewpoint

It is very convenient to introduce a second limit where one “zooms in” on the body as it shrinks. In this limit the external universe becomes unimportant.

First, change to *scaled coordinates*,

$$\bar{t} \equiv \frac{t - t_0}{\lambda}, \quad \bar{x}^i \equiv \frac{x^i - z^i(t_0)}{\lambda}$$

A fixed-xbar observer moves towards the body as it shrinks. Now rescale the metric, charge-current, and stress-energy to keep the body at finite size, mass, and charge. Also rescale the Maxwell field appropriately.

	$\bar{g}_{\mu\nu} \equiv \lambda^{-2} g_{\mu\nu}$	$\bar{J}^{\mu} \equiv \lambda^3 J^{\mu}$	$\bar{T}^{M}_{\mu\nu} \equiv T^{M}_{\mu\nu}$	$\bar{F}_{\mu\nu} \equiv \lambda^{-1} F_{\mu\nu}$
rescale:	size	charge	mass	field

Now the body “looks the same” to our observer as he follows the shrinking body: we have defined the desired zooming in.

Body Parameters in the Near Zone

The worldline Z characterizes the motion at λ=0. For λ>0, however, the body is extended and we need a definition of center of mass. Such a definition must include the energy in the electromagnetic field. However, the slow 1/r falloff of a time-dependent field renders the naïve integral expression divergent. Physically, this is because the integral includes radiation emitted in the arbitrarily distant past!

The near-zone limit provides an escape route valid in perturbation theory to the order we work. At this order, the "zooming in" process has eliminated radiation and the falloff is sufficient to define…

mass $m(t_0) \equiv \int \bar{T}^{(0)}_{00} d^3\bar{x}$ $\delta m(t_0) = \int \bar{T}^{(1)}_{00}\Big|_{\bar{t}=0} d^3\bar{x}$

spin $S^{ij}(t_0) \equiv 2\int \bar{T}^{(0)i}{}_{0}\bar{x}^j d^3\bar{x}$ $S_i = \frac{1}{2}\epsilon_{ijk}S^{jk}$

CM $\bar{X}^i_{\mathrm{CM}}(t_0) = \frac{1}{m}\int \bar{T}^{(0)}_{00}\bar{x}^i d^3\bar{x}$ $\bar{X}^i_{\mathrm{CM}}(t_0) = 0$ defines perturbed worldline

matter

$$\bar{T}_{\bar{\mu}\bar{\nu}} \equiv \bar{T}^{M}_{\bar{\mu}\bar{\nu}} + \bar{T}^{\mathrm{self}}_{\bar{\mu}\bar{\nu}}$$

EM self-field

Where $\bar{x}^i$ gives the spatial distance from the body in its instantaneous rest frame. (That is, these are Fermi normal coordinates.)

Near-zone Computations

Our computational strategy is as follows. Write down the equations arising from conservation of total stress-energy and conservation of charge-current at 0th, 1st, and 2nd order in the near-zone expansions. Multiply these relations by various powers of $\bar{x}^i$ and integrate over space to systematically obtain all relationships holding for the body parameters defined above.

At 0th order we get relationships like that the spin tensor is antisymmetric

At 1st order we obtain:

$$\frac{d}{dt_0} m = 0 \ , \qquad \frac{d}{dt_0} S_{ij} = -Q^{\mu}{}_{[i} F^{\text{ext}}_{j]\mu} \ , \qquad m a_i = q F^{\text{ext}}_{0i}$$

constant mass

the usual dipole torques

Lorentz force

At 2nd order we obtain:

$$\begin{aligned} m\delta a_i &= -(\delta m) a_i + (\delta q) F^{\text{ext}}_{0i} + q\delta F^{\text{ext}}_{0i} + \frac{2}{3} q^2 \dot{a}_i + \\ & \quad + \frac{1}{2} Q^{\mu\nu} \partial_i F^{\text{ext}}_{\mu\nu} + \frac{d}{dt_0} \left(a^j S_{ji} + 2 Q^j{}_{[i} F^{\text{ext}}_{0]j} \right) \\ \frac{d}{dt_0} \delta m &= \frac{1}{2} Q^{\mu\nu} \partial_0 F^{\text{ext}}_{\nu\mu} - \frac{\partial}{\partial t_0} \left(Q^{\mu 0} F^{\text{ext}}_{0\mu} \right) \end{aligned}$$

Note that there is no evolution law for $Q^{\mu\nu}$.

Non-relativistic form of the perturbed force

self-force

dipole energy
effective mass shift

dipole forces

$$\delta\vec{F} \equiv \delta(m\vec{a}) = \frac{2}{3}q^2\frac{d\vec{a}}{dt} + (\vec{p}\cdot\vec{E})\vec{a} + p_i\vec{\nabla}E^i + \mu_i\vec{\nabla}B^i$$

$$+ \frac{d}{dt}\left(\vec{S}\times\vec{a} + \vec{\mu}\times\vec{E} + \vec{p}\times\vec{B}\right)$$

$\vec{p}$: electric dipole moment

$\vec{\mu}$: magnetic dipole moment

$\vec{S}$: spin

Forces arising from “hidden momentum,”
i.e. from p not equaling mV.

The acceleration appearing in the self-force term refers to the <u>background worldline</u>, a solution to the Lorentz force equation that is to be viewed as “already given”.

The equation for the perturbed motion is <u>second-order</u> and <u>no runaway behavior occurs</u>.

Beyond Perturbation Theory

The perturbative corrections to the motion will build up over time, and our perturbative description based on a single, fixed Lorentz force trajectory will be a poor approximation at late times. To improve upon this, we suggest that one should simply invent a “self-consistent perturbative equation” that corrects the Lorentz force trajectory “as one goes along” so as to give a good, global-in-time, description of the motion. (We also maintain that this is done all the time.) The main criteria for the invented equation is that agree with the perturbative result locally in time.

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A first guess at such an equation might be

$$m\vec{a} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) + \frac{2}{3}q^2\frac{d\vec{a}}{dt}$$

which is the ALD equation. However, this equation is the wrong differential order and is therefore unacceptable. (It also has runaway solutions.)

An alternative would be to replace $\vec{a}$ on the RHS with $(q/m)[\vec{E} + \vec{v} \times \vec{B}]$. This “reduced order” equation should be a good description.

Our Viewpoint on EM Self-force

Point particles cannot be consistently coupled to electromagnetism, so one must consider charged continuum matter.

In this context a perturbative result can be rigorously derived, which shows definitively that there is no runaway behavior in classical radiation reaction.

If the perturbative result is good enough in a situation of interest, use it. If a more global-in-time description is required…

Option #1: Use the reduced order ALD equation

Option #2: Use some other self-consistent perturbative equation

Option #3: Evolve perturbatively for a while, then update the background motion; repeat

But the ALD equation is a non-option. It changes the differential order of the system and introduces spurious solutions.

(The result can presumably be applied to elementary particles when they behave classically.)

Results

(the full results)

$$\delta[\hat{m} a_a] = \delta[q F^{\text{ext}}_{ab} u^b] + \left(g_a{}^b + u_a u^b\right) \left\{ \frac{2}{3} q^2 \frac{D}{d\tau} a_b - \frac{1}{2} Q^{cd} \nabla_b F^{\text{ext}}_{cd} + \frac{D}{d\tau} \left(a^c S_{cb} - 2 u^d Q^c{}_{[b} F^{\text{ext}}_{d]c} \right) \right\}$$

$$\frac{D}{d\tau} S_{ab} = 2 \left(g_a{}^c + u_a u^c\right) \left(g_b{}^d + u_b u^d\right) Q^e{}_{[c} F^{\text{ext}}_{d]e} - 2 a^c S_{c[a} u_{b]}$$

$$\frac{D}{d\tau} \delta\hat{m} = \frac{1}{2} Q^{ab} \frac{D}{d\tau} F^{\text{ext}}_{ab} + 2 Q_a{}^b F^{\text{ext}}_{bc} a^{[c} u^{a]}$$

$$\delta\hat{m} \equiv \delta m - u_b u^c Q^{bd} F^{\text{ext}}_{cd}$$

Force in the non-relativistic limit:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) + \frac{2}{3} q^2 \frac{d\vec{a}}{dt} + p_i \vec{\nabla} E^i + \mu_i \vec{\nabla} B^i + \frac{d}{dt}\left(\vec{S} \times \vec{a} - \vec{\mu} \times \vec{E} - \vec{p} \times \vec{B}\right) \; ,$$

If q~e and S~hbar, then “S x a” dominates the self-force by a factor of 137!

A “permanent magnetic dipole”: $p_a = 0, \qquad \mu_a = C S_a$

Then magnetic dipole energy exchanges freely with rest mass: $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m + \mu_a B^a\right) = 0$